ANALYTIC TECHNIQUE FOR HYDRODYNAMIC INSTABILITIES WITH REALISTIC BOUNDARY CONDITIONS

By

KRISHNA KUMAR

DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

JANUARY, 1987

ANALYTIC TECHNIQUE FOR HYDRODYNAMIC INSTABILITIES WITH REALISTIC BOUNDARY CONDITIONS

By
KRISHNA KUMAR

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

JANUARY, 1987

PHY-1987-D-KUM-ANA

To

my

grandmother

and

parents

CERTIFICATE

This is to certify that the work presented in this thesis entitled, "ANALYTIC TECHNIQUE FOR HYDRODYNAMIC INSTABILITIES WITH REALISTIC BOUNDARY CONDITIONS" has been carried out by KRISHNA KUMAR under my supervision. No part of this work has been submitted elsewhere for a degree.

J.K. Bhattacharjee

(Dr.Jayanta K. Bhattacharjee)
Assistant Professor
Department of Physics
Indian Institute of Technology
Kanpur, India

ACKNOWLEDGEMENTS

It is a great pleasure for me to express my deep sense of gratitude and sincere regards to Dr. Jayanta K. Bhattacharjee for introducing me to the vast field of hydrodynamic instability and guiding me through the subject. His active involvement and keen interest in all phases of this work was a source of constant inspiration and path-finding. I offer my sincere thanks to him for his enlightening comments, suggestions and criticisms. My association with him far exceeds the scope of this work, and indeed it has been a rich experience for me.

Words can hardly convey my sense of gratitude to Prof. Kalyan Banerjee for his valuable suggestions and positive criticism of my work from time to time. Many a flaw has been corrected by his critical but positively helpful attitude towards this work.

I am also very much indebted to Dr. G.P. Sastry of IIT, Kharagpur, who provided me with the first authentic glimpse of academic research and gave me constant encouragement since then.

Of the multi-hued spectrum of friends and colleagues who have added spice and strength to my life in this place. I shall be able to name only a few here. They are Prabhat Thakur, Alok Banerjee, Raghava Varma, Roy Saab, Boga, Biplab Ganguli, Manoranjan Prasad Singh, S. Sen, Ramendu Bhattacharya and Doc-Saab, who, through their warmth and concern, generosity,

discussions and encouragement, have made the past few years of my life immensely enjoyable and fruitful. Many others have contributed positively towards the successful completion of this work, and my thanks are due to all of them.

I also thank Mr. U.S. Misra for his patient typing work, and M/S. L.S. Rathaur and H.K. Panda for the neat cyclostyling and compilation of thesis copies.

The warm hospitality which I received from Mrs. U. Bhattacharjee during my stay here will remain a cherished memory with me.

Finally, my heartfelt thanks go to my grandmother, parents and sister for their sustained help and encouragement in all my academic endeavours. I feel deeply indebted to them for whatever I have achieved so far.

KRISHNA KUMAR
IIT Kanpur

CONTENTS

Page

SYNOPSIS

"ANALYTIC TECHNIQUE FOR HYDRODYNAMIC INSTABILITIES WITH REALISTIC BOUNDARY CONDITIONS"

A Thesis Submitted
In Partial Fulfilment of the Requirements
For the Degree of

DOCTOR OF PHILOSOPHY

By

KRISHNA KUMAR

to the

Department of Physics
Indian Institute of Technology, Kanpur

September, 1986

Hydrodynamic instabilities with realistic boundaries are studied analytically in this dissertation. This is necessary for any quantitative comparison between theory and experiment, the latter having become increasingly accurate over the last two decades. The effects of realistic boundaries can be large for unmodulated flows and subtle for modulated ones. Flows are treated in Rayleigh-Bénard (RB) and Couette-Taylor (CT) geometries. Convective instabilities are discussed in both single component and binary fluids in Rayleigh-Bénard configuration.

The traditional boundary conditions assume free bounding surfaces, where the normal component of velocity vanishes and there are no tangential stresses on the fluid. These

constraints make the study of hydrodynamic equations simple. However, these boundary conditions cannot be realized in practice. Realistic boundaries are, in general, rigid and impermeable. At rigid surfaces, not only the normal component, but also the tangential components of velocity vanish because of "no-slip" condition. The impermeability of realistic boundary surfaces prevents any mass current across them. These constraints make the analysis of hydrodynamic systems very difficult. A procedure, due to Chandrasekhar, which makes analytic study with rather general boundary conditions feasible, is followed here.

Chandrasekhar's technique consists of Fourier expanding the field (usually the temperature fluctuation), on which the only requirement is the vanishing at the boundary surfaces, and truncating the expansion at a finite number of terms. With this ansatz on the temperature fluctuation the remaining fields are exactly determined from the differential equations satisfied by them, and the critical Rayleigh number is evaluated self-consistently from the equation for temperature fluctuation. In case of CT flow, the azimuthal velocity fluctuation is expanded and the critical Taylor number is determined from the equation for this fluctuation. The method yields accurate results in the lowest order truncation. The accuracy can be further improved by considering higher order terms of the expansion. In Chapter I of this thesis, a review of the procedure is presented by studying single

component fluid in RB geometry

In Chapter II, the most general binary liquid mixture (^3He-^4He) is studied in RB configuration using this technique. The critical Rayleigh number for the onset of stationary convection is determined in the limit $k/s \gg 1$, where 'k' is the separation parameter and 's' the ratio of mass and thermal diffusivities. The result is within seven percent of the exact numerical value in the lowest order truncation. The truncation of the expansion of temperature fluctuation at the next higher order reduces the error to one percent. While the traditional (free-free) boundary conditions lead to a stationary convection with a non-zero wave number, the realistic boundary conditions cause a stationary convection with zero wave number over a significant range of parameter values.

The procedure is applied to investigate the hydromagnetic convection in RB geometry in Chapter III. The possibility of oscillatory convection is explored in the limit of large Chandrasekhar number ($Q \gg 1$) and small magnetic Prandtl number ($p_2 \ll 1$). It is shown that the Chandrasekhar-Gibson criterion is not a necessary condition for overstability, but a necessary condition for the existence of a polycritical point.

The onset of the first instability in a modulated single fluid in RB configuration is discussed in Chapter IV. The modulation of temperature difference between the two

plates delays the onset of convection, i.e., the correction to the Rayleigh number is positive. At high frequencies, the correction goes to zero as the fifth power of the frequency of modulation.

In Chapter V, this method is used to study the onset of convection in modulated binary liquids. The effect of modulation can be either stabilizing or destabilizing depending upon the parameter values.

The reported analytic and experimental results for modulated CT flow do not agree over the entire range of the frequency of modulation. Hall's analytic study shows destabilization over the whole spectrum of frequencies. In Chapter VI, the modulated CT flow is studied using Chandrasekhar's procedure. The flow is destabilized in the low frequency regime and the results are in good agreement with that of Hall. In the very high frequency regime, the flow is stabilized — a result, qualitatively different from Hall's theoretical work. The correction to Taylor number goes to zero as the fifth power of the frequency of modulation. The procedure, used in the thesis, thus also presents a systematic method for treating the effect of modulation in presence of realistic boundaries.

In the appendix, the onset of oscillatory convection and subsequent period doubling bifurcations in modulated binary liquids are discussed using stroboscopic maps for the system.

OVERVIEW

The onset of hydrodynamic instability in fluids has been a subject of extensive study[1-4]. The recent spate of sophisticated experiments[5-7] has focussed considerable attention on externally modulated fluids[8]. Analytic studies of the modulated systems are limited largely to the case of idealized boundaries, which are physically unrealistic. The results of a recent experiment[6] reveals that even the qualitative features predicted by theory (with idealized boundaries) are not all true for a hydrodynamic system with realistic boundaries. Thus for any quantitative comparison between theory and experiment, the hydrodynamic systems must be analyzed with realistic boundaries. In this dissertation, the onset of hydrodynamic instability in unmodulated as well as modulated flows with realistic boundaries is analytically studied in Rayleigh-Bénard and Couette-Taylor geometries.

Thermal convection in a single-component fluid in Rayleigh-Bénard configuration represents the simplest example of convective instability[9] in fluids. In Rayleigh-Bénard geometry, a thin fluid layer confined between two infinitely extended horizontal plates is heated from below. As the temperature of the fluid is non-uniform throughout its volume, heat energy is conducted from hotter regions to cooler regions and a temperature gradient is maintained

across the fluid layer. On account of positive thermal expansion, hotter parts of the fluid expand and become lighter than cooler parts. The arrangement being 'top-heavy' is unstable, and the temperature gradient is called adverse. The fluid tends to redistribute itself to make the configuration stable (bottom-heavy). This natural tendency is opposed by the viscosity of the fluid. As the adverse temperature gradient is raised by raising the temperature of lower plate, the arrangement becomes more unstable, and for a critical value of the temperature gradient the fluid starts circulating. This causes the onset of thermal convection—an example of hydrodynamic instability. In the case of a single-component fluid, the velocity and temperature fields do not vary with time at the onset of instability; that is, the convection is stationary.

By controlling the temperature difference between the two plates, the adverse temperature gradient can be controlled. So, the temperature difference between the two plates is the 'control parameter' of the problem. The response of the system to a time-periodic disturbance in the control parameter is important. It can lead to drastic modifications in the flow pattern. By varying the temperature of the lower plate periodically, a time-periodic disturbance (an external modulation) can be imposed on the control parameter. The external modulation of the temperature difference between the plates is known to delay the onset of

convection. This means that the external modulation raises the critical value of temperature difference and hence stabilizes the flow-pattern [Chapter IV].

The critical temperature difference depends on the types of bounding surfaces considered. The use of traditional boundary conditions, which assume free bounding surfaces, simplifies the mathematical complexities of theoretical studies. However, for any quantitative comparison with experiments, an analytic study must consider the fluid layer confined between realistic boundaries. Most of the theoretical studies with rigid boundaries use either Galerkin[10,11] or W.K.B.[12] technique. Chandrasekhar's techniques[1] (discussed in Chapter I) makes analytic study with rather general boundary conditions feasible. [redacted]. This procedure yields quite accurate results for various convective as well as centrifugal instabilities in fluids.

A binary fluid[13–20] (a mixture of two nonreactive miscible fluids) or a fluid with a solute (e.g. saline water) is a highly versatile hydrodynamic system for the study of convective instability. The two-component system is characterized by two different diffusivities —— usually those of heat and mass (of the solute) — and, hence, also known as a double-diffusive system. A simultaneous presence of two different diffusion coefficients can

i) drive convection even if the system is hydro-statistically stable[21,22], and

ii) induce either stationary or time-periodic (oscillatory) convection[23] at the onset of of the first instability.

The onset of convection in hydrostatically stable configuration cannot be realized in a single-component fluid. This effect was first anticipated by Stern[21] following an observation by Stommel et al.[22]. Stern[21] considered a thin layer of saline water **with salt concentration** increasing upwards heated from above , the configuration is also known as 'fingering regime'. The adverse concentration gradient tries to destabilize the arrangement, while the favourable temperature gradient tries to stabilize it. For a temperature high enough at the top, the density of the solution decreases upwards making the system hydrostatically stable. Now, if a small parcel of the solution is displaced upwards, it gains heat but not appreciable amount of salt from its surroundings because of small solute diffusivity compared to its heat diffusivity. The parcel becomes hotter and, therefore, lighter. It keeps moving upwards and convective instability occurs in the system. At the onset, convection is always stationary in this configuration.

An oscillatory convection at the onset of hydrodynamic instability was first realized by Veronis[23]. In Veronis configuration — also known as 'diffusing regime' — a thin layer of saline water with its concentration decreasing

upwards is heated from below. If a small parcel of the solution is displaced upwards, it loses heat but not salt. It becomes heavier as it cools, and the buoyancy force drives it back towards its initial position. The parcel starts oscillating. A lag in temperature between the parcel and its surroundings makes the amplitude of oscillation grow in time provided viscosity of the solution is not too large. This means that the parcel returns to its mean position of rest faster than it leaves. The oscillatory instability thus produced is called overstable convection. The instability is generally oscillating for saline water in this configuration. For other solutions, the first instability can be stationary also. These effects can occur also in a binary fluid mixture. It was first demonstrated by Schechter et al.[14]. In fact, a system of binary liquids is more advantageous than salt-water mixture for experimental studies. In case of binary liquids, particularly ^3He-^4He mixture, the ratio of the two diffusivities can be parametrically varied over a wide range by varying the temperature difference between the plates. It is very difficult to do the same experimentally for a salt-water mixture.

Most of the theoretical studies on binary liquids are confined to idealized (free) boundary conditions. The effects of realistic (rigid and impermeable) boundary conditions can be drastic. While the free boundaries cause stationary convection

with non-zero wave number, realistic boundaries lead to stationary convection of zero wave number over a significant range of parameter values [Chapter II]. If an external modulation is imposed on the temperature difference between the bounding surfaces, the onset of stationary convection can either be delayed or preponed depending upon the parametric values. That is, the modulation can either stabilize or destabilize the flow pattern in binary fluids depending upon the parametric range. In case of oscillatory convection in binary fluids, the modulation always destabilizes the ~~flow pattern~~ [Chapter V].

If a single-component fluid enclosed in Rayleigh-Bénard configuration is electrically conducting, an externally imposed magnetic field delays the onset of thermal instability. At the onset the convection is generally stationary. The possibility of a time-periodic (oscillatory) convection in this system is a problem of long standing. Chandrasekhar[1] studied this problem using idealized boundary conditions and showed that the magnetic Prandtl number (p_2) had to be larger than the thermal Prandtl number (p_1) for any possibility of oscillatory convection at all. However, under terrestrial conditions $p_2 << p_1$. Later, Gibson[24] analyzed this hydromagnetic system with realistic boundary conditions and obtained Chandrasekhar's criterion once again. Recently Banerjee et.al.[25] predicted the possibility of overstable convection — oscillatory convection with growing amplitude, for $p_2 < p_1$ but in the presence of large magnetic field. This

demands re-examination of the flow with proper boundary conditions. In Chapter III, it is shown that Chandrasekhar-Gibson criterion is not a necessary condition for the onset of oscillatory condition, but rather a necessary condition for the existence of a polycritical point — a point in parameter space where the line of oscillatory convection meets the line of stationary convection.

Couette-Taylor flow is the simplest system to study the hydrodynamic instability in rotating fluids. In this arrangement, a fluid is enclosed between two infinitely long coaxial vertical cylinders. The inner one of them is rotating with a constant angular velocity (Ω), while the outer one is at rest. Since the inner layers are rotating faster than the outer ones, the fluid tends to move radially. This is counteracted by a pressure gradient developed because of viscous drag. There is no vertical component of the velocity for low rotational speed because of the symmetry of the system. Thus the basic flow, also known as 'Couette flow', is always tangential to the cylinders. As 'Ω' is increased, the centrifugal forces on inner layers grow, and for a critical Ω ($= \Omega_c$) they become larger than the viscosity induced pressure gradient. As the inner layers cannot move outward uniformly because of the outer layer in their ways, they break into bands and circulation starts. These bands are called 'Taylor Vortices' and the transition from Couette flow is termed as 'Couette-Taylor instability'. At the onset of instability,

the flow pattern is stationary. Further enhancement of 'Ω' makes Taylor vortices wavy and finally leads to turbulence in the fluid.

The reported theoretical and experimental results for modulated Couette-Taylor[26-30] flow do not agree over the entire range of the frequency of modulation. The full hydrodynamic equations for this system have been studied analytically by Hall[26] and numerically by Riley and Laurence[27]. While Hall finds destabilization of flow due to modulation at all frequencies, Riley and Laurence[29] notice that, for large amplitudes of modulation, the effect of modulation is to destabilize the flow for low frequencies and stabilize it for high frequencies. For small amplitudes, Riley and Laurence do not notice the stabilization at high frequencies perhaps because the effect at high frequencies is very small and barely within the range of numerical accuracy for moderate amplitude and beyond it for smaller amplitudes. The experiments of Donnelly[29] show a stabilization at all frequencies, while the more accurate experiments of Thompson[30] show a destabilization at low frequencies and a stabilization at high frequencies. In Chapter VI of the thesis we analyze the problem with realistic boundary conditions using Chandrasekhar's technique[1], and show that the effect of modulation (small amplitude modulation) is to destabilize the flow at low frequencies and to stabilize at high frequencies — a result qualitatively different from that

of Hall[26].

A low-order truncation[31-34] facilitates the handling of non-linearity in hydrodynamic equations. In the appendix, a stroboscopic map[35] is constructed from Lorenz-like truncated equations for binary liquids and the possibility of period-doubling chaos[36-37] is discussed.

REFERENCES

1. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability" (Clarenden Press, Oxford 1961).

2. Lin, C.C., "The Theory of Hydrodynamic Stability" (University Press, Cambridge 1955).

3. Joseph, D.D., "Stability of Fluid Motions I and II" (Springer, Berlin, Heidelberg, New York, 1977).

4. Swinney, H.L. and Gollub, J.P., "Hydrodynamic Instabilities and the Transition to Turbulence" (Springer, Berlin, New York 1981).

5. Rehberg, I. and Ahlers, G., Phys. Rev. Lett. 55, 500. (1985).

6. Ahlers, G. and Rehberg, I., Phys. Rev. Lett. 56, 1373 (1986).

7. Gao, H. and Behringer, H., Phys. Rev. A34, 697 (1986).

8. Davis, S.H., Ann. Rev. Fluid Mech. 8, 57 (1976).

9. Normand, C., Pomeau, Y. and Velarde, M., Rev. Mod. Phys. 49, 581 (1977).

10. Gresho, P.M. and Sani, R.L., J. Fluid Mech. 40, 783 (1970).

11. Rosenblat, S. and Tanaka, G.A., Phys. Fluids 14, 1319 (1971).

12. Rosenblat, S. and Herbert, D.M., J. Fluid Mech. 43, 385 (1970).

13. Hurle, D.T.J. and Jackeman, E., Phys. Fluids. 12, 2704 (1969) and J. Fluid Mech. 47, 667 (1971).

14. Schechter, R.S., Prigogine, I. and Hamm, J.R., Phys. Fluids 15, 379 (1972).

15. Legros, J.C., Platten, J.K. and Poty, P., Phys. Fluids 15, 1383 (1972).

16. Velarde, M.G. and Schechter, R.S., Phys. Fluids. 15, 1707 (1972).

17. Platten, J.K. and Legros, J.C., Phys. Rev. Lett. 45, 1944 (1980).

18. Knobloch, E., Phys. Fluids 23, 1981 (1980).

19. Platten, J.K. and Legros, J.C., "Convection in Liquids". Springer, NY (1984), Ch. IX.

20. Behringer, R.P., Rev. Mod. Phys. 57, 657 (1985).

21. Stern, M.E., Tellus 12, 172 (1962).

22. Stommel, H., Arons, A. and Blanchard, D., Deep Sea Res. 3, 152 (1956).

23. Veronis, G., J. Mar. Res. 23, 1 (1965).

24. Gibson, R.D., Proc. Cambridge Philos. Soc. 62, 287 (1966).

25. Banerjee, M.B., Gupta, J.R., Shandil, R.G., Sood, S.K., Banerjee, B. and Banerjee, K., J. Math.Anal. Appl. 108, 216 (1985).

26. Hall, P., J. Fluid Mech. 67, 29 (1975).

27. Riely, P.J. and Laurence, R.L., J. Fluid Mech. 75, 625 (1976).

28. Meister, B., and Muzner, W., Angew, Z., Math. Phys. 17, 537 (1966).

29. Donnelly, R.J., Proc. Roy. Soc. A281, 130 (1964).

30. Thompson, R., Ph.D. Thesis, Department of Meteorology, MIT (1968).

31. Lorenz, E., J. Atoms, Sci. 20, 130 (1963).

32. Mclaughlin, J.B.and Mertic, P.C., Phys. Rev. A12, 186 (1975).

33. DaCosta, L.N., Knobloch, E. and Weiss, N.O., J. Fluid Mech. 109, 25 (1982).

34. Agarwal, A., Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. B30, 6458 (1984).

35. Broomhead, D., McCreadie, G. and Rowlands, G., Phys. Letts. 84A, 229 (1986).

36. Feigenbaum, M.J., J. Stat. Phys. 19, 25 (1978).

37. Bai-Lin, H., "Chaos" [World Scientific,Singapore]1984 .

CHAPTER I

A REVIEW OF CHANDRASEKHAR'S TECHNIQUE

1.1 Introduction

The advent of sophisticated experimental techniques[1-3] has provoked considerable attention recently in the study of hydrodynamic instabilities[4-7]. Recent experimental results[2] show that even the qualitative features predicted by theory are not all shared by the hydrodynamic system with realistic boundaries. Thus to obtain any reliable results, the effects of realistic boundary conditions must be included in the theoretical studies of hydrodynamic instability. The consideration of "no-slip" condition because of the rigidity of realistic boundaries makes the analysis very difficult, particularly in presence of an external modulation.

The effect of realistic boundaries on both modulated and unmodulated hydrodynamic flows cam be studied by using Chandrasekhar's[4] technique. In this chapter, we review the technique by studying the convective instability in single-component fluid in Rayleigh-Bénard geometry. The instability in the modulated flow has been studied in Chapter IV.

1.2 Hydrodynamic Problem

An incompressible fluid of viscosity 'μ' is confined between two rigid horizontal plates of infinite extension. The plates are a distance 'd' apart. The lower plate is at a temperature T_1, while the upper one is at T_2 (< T_1). In conduction state there is no motion in the fluid. This means that the force due to gravity is balanced by the pressure gradient. Steady state pressure field is determined from the equation

$$\frac{\partial P_s}{\partial z} = \rho_s \, g \quad , \tag{1.1}$$

and the conduction of heat in the same state is expressed as

$$\frac{\partial^2 T_s}{\partial z^2} = 0 \quad . \tag{1.2}$$

The boundary conditions on temperature field determines T_s, which found to be

$$T_s = T_m - \Delta T \left(\frac{z}{d}\right) \quad , \tag{1.3}$$

where T_m $\left(= \frac{T_1 + T_2}{2}\right)$ is the mean temperature, ΔT (= $T_1 - T_2$), the temperature difference between the plates. The co-ordinate system is such that the plates are normal to z-axis at positions z = +1/2 and z = −1/2.

As we raise the temperature difference (ΔT), for a critical value ΔT_c, convection starts. At the onset, the velocity field ($\vec{V}$) can be assumed small. Because of the

instability all the fields will be disturbed from their steady state values. If $\theta, \delta\rho$ and δP denote the fluctuations in temperature field, density and pressure field, then the onset of instability, they can be written as

$$T = T_s + \theta \ , \qquad (1.4a)$$

$$\rho = \rho_s + \delta\rho \ , \qquad (1.4b)$$

$$P = P_s + \delta P \ . \qquad (1.4c)$$

The heat transfer is now governed by the equation

$$\frac{\partial T}{\partial t} + \vec{V}.\vec{\nabla}T = \lambda \nabla^2 T \ , \qquad (1.5)$$

where λ is the thermal diffusivity. The vector $\vec{X}$ and the velocity field $\vec{V}(\vec{X},t)$ have components (x,y,z) and (u,v,w) respectively. The incompressibility prevents divergence of velocity field, that is,

$$\vec{\nabla}.\vec{V} = 0 \ . \qquad (1.6)$$

The fluid motion is described by Navier-Stokes equations, which are

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{V} = -\frac{\vec{\nabla}P}{\rho_m} + \nu\nabla^2 V + \vec{g}\frac{\rho}{\rho_m} \qquad (1.7)$$

where ρ_m is the mean density at the mean temperature T_m, $\nu \ (= \frac{\mu}{\rho_m})$, the kinematic viscosity.

The equation of state is

$$\rho = \rho_m \left[1 - \bar{\alpha}(T - T_m)\right] \ , \qquad (1.8)$$

where

$$\bar{\alpha} = - \frac{1}{\rho_m} \left(\frac{\partial \rho}{\partial T}\right)$$

is the coefficient of thermal expansion.

In deriving Eqs.(1.5)-(1.7), Bossinesq approximation is applied. For variations in the temperature upto 10°K, the variations in the density are negligible because of the smallness of 'α'. The other coefficients (ν, λ) can be assumed constant. But the force of buoyancy due to variation in the density cannot be neglected. The magnitude of the buoyancy force is

$$(\delta\rho)\, g = (\rho - \rho_m) g = \rho_m \alpha (\Delta T) g, \qquad (1.9)$$

and this can be quite larger than other terms. Thus ρ (or ρ_s) is assumed a constant equal to ρ_m in all the terms of Navier-Stokes Eq.(1.7), except the term representing the force due to gravity.

The linearized hydrodynamical equations for small disturbances are

$$\frac{\partial \vec{V}}{\partial t} = - \frac{\vec{\nabla}\delta P}{\rho_m} - g\alpha\theta\, \hat{z} + \nu\nabla^2 \vec{V}, \qquad (1.10)$$

$$\frac{\partial \theta}{\partial t} + (\vec{V}.\vec{\nabla} T_s) = \lambda\nabla^2 \theta, \qquad (1.11)$$

where

$$\delta\rho = -\alpha\rho_m \theta \text{ and } \vec{g} = -g\hat{z} \qquad (1.12)$$

are used.

The boundary conditions on temperature fluctuation are

$$\theta = 0 \quad \text{at } z = \pm 1/2 .$$

At the rigid surfaces, not only the vertical component, but also the tangential components of the velocity field vanish because of 'no-slip' condition. Using of this fact in Eq.(1.6) yields

$$\frac{\partial w}{\partial z} = 0 . \tag{1.13}$$

The term proportional to $\vec{\nabla}\delta p$ in Eq.(1.7) is eliminated by taking curl of this equation twice. The z-component of the resulting equation is

$$\frac{\partial}{\partial t} \nabla^2 w = g\alpha \nabla_1^2 \theta + \nu \nabla^2 w, \tag{1.14}$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \tag{1.15}$$

To see the stability of the flow, the response of the system to all possible disturbances has to be investigated. This is done by expressing any arbitrary disturbance as a superposition of possible normal modes and examining the stability of the system with respect of each of them. For the geometry considered here, the normal component (z-component) of velocity field and temperature fluctuation can be written in terms of two-dimensional periodic waves as given below:

$$w = W(z,t)\, e^{i(k_x x + k_y y)} \tag{1.16}$$

$$\theta = \Theta(z,t)\, e^{i(k_x x + k_y y)} , \tag{1.17}$$

where $\bar{k} = \sqrt{k_x^2 + k_y^2}$ is the two-dimensional wave number associated with the disturbances $W(z,t)$ and $\Theta(z,t)$. Using Eqs.(1.16), (1.17) and (1.3) in Eqs.(1.14) and (1.11),

$$\frac{\partial}{\partial t}\left(\frac{d^2}{dz^2} - k^2\right) W = -g\alpha k^2 \Theta + \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 W \quad (1.18)$$

and

$$\frac{\partial \Theta}{\partial t} = W\left(\frac{\Delta T}{d}\right) + \lambda\left(\frac{d^2}{dz^2} - k^2\right)\Theta . \quad (1.19)$$

For convenience, the system of hydrodynamical equations is non-dimensionalized. The length is measured in the units of the separation 'd' between the plates, the time in units of 'd^2/ν'. The non-dimensional wave number is defined as $a = (\bar{k}d)$. Then Eqs.(1.18) and (1.19) become

$$(D^2-a^2)(D^2-a^2 - \frac{\partial}{\partial t}) W = \left(\frac{g\,\alpha d^2}{\nu}\right) a^2 \Theta \quad (1.20)$$

and

$$(D^2-a^2 - p_1 \frac{\partial}{\partial t}) \Theta = -\left(\frac{\Delta T}{\lambda} d^2\right) W , \quad (1.21)$$

where,

$p_1 \left(= \frac{\nu}{\lambda}\right)$ is the Prandtl number

and

$$D \equiv \frac{d}{dz}, (z \rightarrow z/d) .$$

At the onset of stationary convection, the time dependence can be ignored in Eqs.(1.20) and (1.21). Thus, the relevant perturbation equations in non-dimensional form are

$$(D^2 - a^2)^2 W = \Theta \tag{1.22}$$

and

$$(D^2 - a^2)\Theta = -Ra^2 W \tag{1.23}$$

with boundary conditions

$$\Theta = W = DW = 0 \text{ at } z = \pm 1/2. \tag{1.24}$$

1.3 Chandrasekhar's Technique

The system of differential Eqs.(1.22) - (1.23) with boundary conditions (1.24) is solved using the following procedure.

The temperature fluctuation field (Θ), on which the only requirement is the vanishing at the surfaces at $z = \pm 1/2$, is expanded in Fourier cosine series as

$$\Theta = \sum_{n=0}^{\infty} A_n \cos(2n+1)\pi z. \tag{1.25}$$

The above expansion is inserted in Eq.(1.22). This gives

$$(D^2-a^2)^2 W = \sum_n A_n \cos(2n+1)\pi z. \tag{1.26}$$

The general solution of this equation is

$$W = B\cosh(az) + Cz\sinh(az) + \sum_n \frac{A_n \cos(2n+1)\pi z}{\{(2n+1)^2\pi^2 + a^2\}^2}. \tag{1.27}$$

Constants B and C are determined from the constraints on W. They are found to be

$$2B = -C \tanh(a/2) \quad . \tag{1.28}$$

and

$$C = \frac{2}{(\sinh a + a)} \sum_n \frac{(-)^n (2n+1)\pi A_n}{\{(2n+1)^2 \pi^2 + a^2\}^2} \cosh(a/2). \tag{1.29}$$

Inserting Eqs.(1.25) and (1.27) in (1.23) yields

$$(D^2-a^2) \sum_n A_n \cos(2n+1)\pi z = -Ra^2 \Big[B\cosh(az) + Cz \sinh(az) + \\ + \sum_n \frac{A_n \cos(2n+1)\pi z}{\{(2n+1)^2\pi^2 + a^2\}^2} \Big],$$

which implies

$$\{(2n+1)^2\pi^2 + a^2\} A_n \cos(2n+1)\pi z = Ra^2 \Big[B \cosh(az) + Cz \sinh(az) \\ + \sum_n \frac{A_n \cos(2n+1)\pi z}{\{(2n+1)^2\pi^2 + a^2\}^2} \Big]. \tag{1.30}$$

Multiplying throughout with $\cos(2n+1)\pi z$ and integrating over the entire range of $z(z = -1/2$ to $z = +1/2)$, we find

$$\Big[\{(2n+1)^2\pi^2 + a^2\} - \frac{Ra^2}{\{(2n+1)^2\pi^2 + a^2\}^2} \Big] A_n \\ = 2Ra^2 B \int_{-1/2}^{+1/2} \cosh(az) \cos(2n+1)\pi z \; dz \\ + 2Ra^2 C \int_{-1/2}^{+1/2} z \sinh(az) . \cos(2n+1)\pi z \; dz \; . \tag{1.31}$$

Performing integrations and using Eqs. (1.28) and (1.29),

$$[\{(2n+1)^2\pi^2+a^2\} - \frac{Ra^2}{\{(2n+1)^2\pi^2 + a^2\}^2}] A_n$$

$$= - Ra^2 [\frac{16a}{(sinha + a)} \frac{(2n+1)^2\pi^2 A_n \cosh^2(a/2)}{\{(2n+1)^2\pi^2 + a^2\}^4}$$

$$+ \frac{16a \cosh(a/2)}{(\sinh a+a)} \sum_{m\neq n} \frac{(-)^{m+n}(2m+1)(2n+1)\pi^2 A_n}{\{(2m+1)^2\pi^2+a^2\}^2\{(2n+1)^2\pi^2+a^2\}^2}] \tag{1.32}$$

Now truncating the expansion (1.25) at the first term (n=0),

$$(\pi^2+a^2)^3 = R^{(o)}a^2 [1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2+a^2)^2(\sinh a + a)}],$$

which results in

$$R^{(o)} = \frac{(\pi^2 + a^2)^3/a^2}{[1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2+a^2)^2(\sinh a+a)}]}, \tag{1.33}$$

To find critical value of $R^{(o)}$ $(= R_c^o)$, at which the first convective instability occurs, $R^{(o)}(a)$ in Eq.(1.33) is minimized with respect to 'a'. This yields $a_c = 3.12$ and

$R_c^{(o)}$ = 1712, within 1% of the exact numerical results. Thus, the technique in the lowest order produce quite accurate results. The accuracy can be further improved by considering higher order terms in the expansion of temperature fluctuation field.

REFERENCES

1. Rehberg, I. and Ahlers, G., Phys. Rev. Lett. 55 , 500. (1985).

2. Ahlers, G. and Rehberg, I., Phys. Rev. Lett. 56, 1373 (1986).

3. Gao, H. and Behringer, R., Phys. Rev. A34, 697 (1986).

4. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability" (Clarendon Press, Oxford 1961).

5. Lin, C.C., "The Theory of Hydrodynamic Stability" (University Press, Cambridge 1955).

6. Joseph, D.D., "Stability of Fluid Motions I and II" (Springer, Berlin, Heidelberg, New York 1976).

7. Swinney, H.L. and Gollub, J.P., "Hydrodynamic Instabilities and the Transition to Turbulence" (Springer-Verlag, Berlin, Heidelberg, New York 1981).

CHAPTER II

CONVECTION IN BINARY LIQUIDS

2.1 Introduction

The convection in a binary liquid[1-13] — a mixture of two completely miscible and non-reactive liquids — is more versatile than the single-component Rayleigh-Bénard convection because of the possibility of both stationary and oscillatory instabilities at the onset. A mixture of ^{3}He-^{4}He near the superfluid transition temperature (T_λ) is the most general binary liquid mixture from the experimental point of view because of the following reasons:

i) the mass diffusion of ^{3}He into ^{4}He increases rapidly near the lamda point (superfluid transition temperature) and ultimately diverges at the transition point[14], while the thermal conductivity at zero mass current remains finite;

ii) the thermodiffusion coefficient, unlike that of any other binary liquid, becomes large and reaches a limiting value[15] of about 0.57 at the lamda point; and the effect of concentration gradient producing a heat current no longer be ignored;

iii) the coefficient of thermal expansion changes sign near the lamda point[16].

Because of the striking properties (i) and (ii), the ratio of mass diffusivity to the thermal conductivity (in the absence of mass current) in the $^3He-^4He$ mixture can be parametrically varied over a wide range - from a number much less than unity away from T_λ to a number much greater than unity very close to T_λ . Experimental advantages like good temperature stability and resolution provide a good control over the temperature field. Thus by varying the mean temperature of the liquid mixture all different possibilities at the onset and subsequent transitions[17-20] can be investigated.

The onset of convection in binary liquids using rigid boundary conditions is discussed in depth by Gutkowiz-Krusin et al.[21] They use a variational principle for stationary instability and wavefunctions similar to the variational ones for treating the oscillatory instability for which there is no variational principle. In this chapter, the onset of convective instability — both stationary and oscillatory — with realistic boundary conditions has been studied using Chandrasekhar's technique[22].

2.2 Hydrodynamics of Binary Liquids

Hydrodynamic equations for the temperature (T) and concentration (c) variables in the binary mixture at the

onset have been obtained by Landau-Lifshitz[23] as

$$\frac{\partial T}{\partial t} + (\vec{V}.\vec{\nabla}T) = \lambda \nabla^2 T \quad , \tag{2.1}$$

$$\frac{\partial c}{\partial T} + (\vec{V}.\vec{\nabla}c) = \mathcal{D}\left[\nabla^2 c + \frac{k_T}{T_m} \nabla^2 T\right], \tag{2.2}$$

because the heat current, due to concentration gradient, is negligibly small in most of the fluids.

Here, $\vec{V}(\vec{X}, t)$ is the velocity of an elementary volume of the liquid mixture at any point $\vec{X}$ in the system. The concentration 'c' is the mass fraction of ^{3}He, $\mathcal{D}$, the isothermal mass diffusion coefficient, λ, the thermal diffusivity in the absence of concentration gradient and k_T, the thermodiffusion coefficient.

In Boussinesq approximation, the Navier-Stokes equations describing velocity field in the liquid are given by

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\, \vec{V} = - \frac{1}{\rho_m} \vec{\nabla}P + \vec{g}\, \frac{\rho}{\rho_m} + \nu\nabla^2\, \vec{V}, \tag{2.3}$$

where ρ_m is the mean density, ρ, the position dependent density and ν, the kinematic viscosity. The incompressibility of the fluid is expressed as

$$\vec{\nabla} . \vec{V} = 0 \, . \tag{2.4}$$

The equation of state is

$$\rho = \rho_m \left[1 - \bar{\alpha}(T - T_m) - \bar{\beta}(c - c_m) \right], \tag{2.5}$$

where T_m and c_m are the mean temperature and concentration

respectively. $\bar{\alpha}$ is the coefficient of thermal expansion

$$\bar{\alpha} = - \frac{1}{\rho_m} \frac{\partial \rho}{\partial T} , \tag{2.6a}$$

and
$$\bar{\beta} = - \frac{1}{\rho_m} \frac{\partial \rho}{\partial c} , \tag{2.6b}$$

measures the change in density of the mixture with change in ^3He concentration. $\bar{\beta}$ is positive according to the definition.

In the steady conduction state, there is no motion in the fluid, i.e., $\vec{V} = \vec{V}_s = 0$ and the temperature and concentration fields are governed by the different equations

$$\frac{\partial^2 T_s}{\partial z^2} = 0 \tag{2.7a}$$

and
$$\frac{\partial^2 c_s}{\partial z^2} = 0 , \tag{2.7b}$$

where the z-axis is positive in vertically upward direction. If T_1, c_1 and T_2, c_2 are temperatures , concentrations (of one component) of the mixture at the lower and upper plates respectively, steady state profiles for those variables are

$$T_s(z) = T_1 - (\Delta T) \frac{z}{d} , \tag{2.8a}$$

$$c_s(z) = c_1 - (\Delta c) \frac{z}{d} , \tag{2.8b}$$

where $\Delta T = T_1 - T_2$ and $\Delta c = c_1 - c_2$.
As there is no mass current in the steady state,

$$\Delta c + \frac{k_T}{T_m} \Delta T = 0$$

that is,
$$\frac{\Delta c}{\Delta T} = - \frac{k_T}{T_m} . \tag{2.9}$$

If $\delta\rho$, δT, δc and δP be the fluctuations from steady state values for density, temperature, concentration and pressure because of the onset of instability, then hydrodynamic equation for these fluctuations are

$$\frac{\partial \vec{v}}{\partial t} = - \frac{\vec{\nabla}(\delta P')}{\rho_m} - \left(\frac{\delta\rho}{\rho_m}\right) g \hat{z} + \nu \nabla^2 \vec{v}, \tag{2.10}$$

$$\frac{\partial(\delta T)}{\partial t} + w \frac{\partial T_s}{\partial z} = \lambda \nabla^2 (\delta T), \tag{2.11}$$

$$\frac{\partial(\delta c)}{\partial t} + w \frac{\partial c_s}{\partial z} = \mathcal{D} \nabla^2 (\delta c) + \mathcal{D} \frac{k_T}{T_m} \nabla^2 (\delta T), \tag{2.12}$$

The term $\vec{\nabla}(\delta P)$ in Eq.(2.10) is eliminated by taking curl of the equation twice, as done in Chapter I. The z-component of the resulting equation is

$$\frac{\partial}{\partial t} \nabla^2 w = g \alpha \nabla_1^2 (\delta T) + \nu \nabla^4 w , \tag{2.13}$$

with $\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Now, Eqs.(2.11)-(2.13) are analyzed in normal modes by seeking solutions of the form

$$\delta T = \Theta(z,t) . e^{i(k_x x + k_y y)} , \tag{2.14a}$$

$$\delta c = C(z,t) . e^{i(k_x x + k_y y)} , \tag{2.14b}$$

$$w = W(z,t) . e^{i(k_x x + k_y y)} , \tag{2.14c}$$

where $\bar{k} = \sqrt{k_x^2 + k_y^2}$, is the wave number of the convection rolls.

By using Eqs.(2.14a) - (2.14c) and Eqs.(2.8a)-(2.8b) in Eqs.(2.10)-(2.12) and linearizing the resulting equations, they can be written in non-dimensional form as

$$(D^2-a^2)(D^2 - a^2 - p)\, W = Ra^2\, (\Theta - k\, C), \tag{2.15}$$

$$(D^2-a^2 - \sigma p)\Theta = -W\,, \tag{2.16}$$

$$(D^2-a^2 - \frac{\sigma}{s}\, p)C = (D^2 - a^2)\Theta, \tag{2.17}$$

where p is the growth rate of fluctuations, $\sigma(= \frac{\nu}{\lambda})$ the thermal Prandtl number, $R\,[\, = \frac{\bar{\alpha}\, g(\Delta T)\, d^3}{\lambda \nu}]$ the Rayleigh number, s the ratio of mass diffusivity ($\mathcal{D}$) to thermal diffusivity (λ), $k(= \frac{k_T}{T_m}\, \frac{\bar{\beta}}{\bar{\alpha}})$ the separation parameter. In Eqs.(2.15)-(2.17), the length is measured in units of separation (d) between the two plates, time, in units of d^2/ν, θ, in (ΔT), and C, in (Δc). $a = (\bar{k}\, d)$

Defining a new variable, the fluctuation in mass current

$$\delta j = (\delta c + \frac{k_T}{T_m}\, \delta T) = \Delta c\, [\frac{\delta c}{\Delta c} + (\frac{\Delta T}{\Delta c})\, \frac{k_T}{T_m}\, (\frac{\delta T}{\Delta T})] = \Delta c\, (C - \Theta), \tag{2.18}$$

and

$$J = \frac{\delta j}{\Delta c} = (C - \Theta), \tag{2.19}$$

Now, from Eqs.(2.5) and (2.19),

$$\frac{\delta\rho}{\bar{\rho}_m} = \bar{\alpha}\Delta T \, [(1-k)\Theta - kJ] \, . \qquad (2.20)$$

Using Eq. (2.19) in Eqs.(2.15)-(2.17), the hydrodynamic equations are transformed as follows:

$$(D^2-a^2)(D^2-a^2-p)W = Ra^2(1-k)\Theta - Ra^2 kJ, \qquad (2.21)$$

$$(D^2-a^2-\sigma p)\,\Theta = -W \, , \qquad (2.22)$$

$$(D^2 - a^2 - \frac{\sigma}{s} p)\,J = (D^2 - a^2)\,\frac{\Theta}{s} \qquad (2.23)$$

2.3 <u>Boundary Conditions</u>

(i) Rigid bounding surfaces imply

$$W = DW = 0 \text{ at } z = \pm\, 1/2 \qquad (2.24a)$$

(ii) Since, the temperatures of lower and upper plates are fixed and therefore,

$$\Theta = 0 \text{ at } z = \pm\, 1/2 \qquad (2.24b)$$

(iii) Impermeability prevents any mass current across the boundaries, that is,

$$DJ = 0 \text{ at } z = \pm\, 1/2 \qquad (2.24c)$$

2.4 <u>Stationary Convection</u>

In this case, the fluctuations are time-independent and, hence, $p = 0$. Now, following Chandrasekhar's procedure, the symmetric solution to the temperature fluctuation, Θ, is

expanded in Fourier cosine series as

$$\Theta = \sum_{n=0}^{\infty} A_n \cos(2n+1)\pi z, \tag{2.25}$$

which is consistent with the boundary conditions on 'Θ'. Inserting the expansion for temperature fluctuation from Eq.(2.25) in Eq.(2.23) with the proper boundary conditions on mass current [Eq.(2.24c)], fluctuation in mass current is determined exactly. It is found to be

$$J = \frac{1}{s}\left[\sum_n A_n \cos(2n+1)\pi z + \frac{\pi}{a} \sum_n (-)^n A_n (2n+1) \frac{\cosh(az)}{\sinh(a/2)}\right]. \tag{2.26}$$

Operating by (D^2-a^2) on Eq.(2.21) and using Eq.(2.23) with $p = 0$, we find

$$(D^2-a^2)^3 W = Ra^2(1-k-\frac{k}{s})(D^2-a^2)\Theta. \tag{2.27}$$

The additional conditions on the z-component of velocity field are

$$(D^2-a^2)^2 W = -kRa^2 J \text{ at } z = \pm 1/2 . \tag{2.28}$$

Inserting Eq.(2.25) in Eq.(2.27), the general solution of the latter is found to be

$$W(z) = B_1 \cosh(az) + B_2 z \sinh(az) + B_3 z^2 \cosh(az) + Ra^2 \sum_n \frac{A_n (1-k-\frac{k}{s})}{\{(2n+1)^2\pi^2 + a^2\}^2} \cos(2n+1)\pi z . \tag{2.29}$$

The boundary conditions in Eq.(2.28) yields

$$B_3 = -\frac{kR}{8s} \cdot \frac{1}{\sinh(a/2)} \cdot \frac{\pi}{a} \sum_n (-)^n (2n+1) A_n . \tag{2.30}$$

The boundary conditions $W = DW = 0$ at $z = \pm 1/2$ give

$$B_1 \cosh(a/2) + \frac{B_2}{2} \sinh(a/2) = -\frac{B_3}{4} \cosh(a/2) \tag{2.31a}$$

and

$$a B_1 \sinh(a/2) + B_2 [\sinh(a/2) + \frac{a}{2} \cosh(a/2)]$$

$$= -B_3 [\frac{a}{4} + \coth(a/2)] \sinh(a/2)$$

$$+ Ra^2 (1 - k - \frac{k}{s}) \sum_n \frac{(-)^n (2n+1) A_n}{\{(2n+1)^2 \pi^2 + a^2\}^2} . \tag{2.31b}$$

B_1 and B_2 are obtained from the above two equations and are

$$B_2 = \frac{2}{(\sinh a + a)} [\alpha_1 \sum_n P_n + \beta_1 \sum_n Q_n \coth(a/2)] \cosh(a/2) , \tag{2.32a}$$

and

$$B_1 = -\frac{1}{2} [B_2 \coth(a/2) + \frac{B_3}{2}] , \tag{2.32b}$$

where

$$\alpha_1 = Ra^2 (1 - k - \frac{k}{s}), \tag{2.33a}$$

$$\beta_1 = \frac{kR}{8s} \cdot \frac{\pi}{a} , \tag{2.33b}$$

$$P_n = \frac{(-)^n (2n+1)\pi A_n}{\{(2n+1)^2\pi^2 + a^2\}^2}, \qquad (2.33c)$$

and

$$Q_n = (-)^n (2n+1) A_n . \qquad (2.33d)$$

Turning now to the differential Eq.(2.22) [with p = 0] and using the solutions for Θ and W, it can be written as

$$(D^2-a^2) \sum_n A_n \cos(2n+1)\pi z = - B_1 \cosh(az) - B_2 z \sinh(az) -$$

$$- B_3 z^2 \cosh(az) - \alpha_1 x$$

$$x \sum_n \frac{A_n}{\{(2n+1)^2\pi^2 + a^2\}^2} \cos(2n+1)\pi z .$$

(2.34)

Multiplying throughout with $\cos(2n+1)\pi z$ and integrating from $z = -1/2$ to $z = +1/2$, the above equation yields

$$[(2n+1)^2\pi^2 + a^2] A_n = \frac{\alpha_1 A_n}{\{(2n+1)^2\pi^2 + a^2\}^2} + 2B_1 x$$

$$x \int_{-1/2}^{+1/2} \cosh(az) \cos(2n+1)\pi z \; dz +$$

$$+ 2B_2 \int_{-1/2}^{+1/2} z \sinh(az) . \cos(2n+1)\pi z \; dz +$$

$$+ 2B_3 \int_{-1/2}^{+1/2} z^2 \cosh(az) . \cos(2n+1)\pi z \; dz .$$

(2.35)

Performing the integrations,

$$[\{(2n+1)^2\pi^2 + a^2\} - \frac{\alpha_1}{\{(2n+1)^2\pi^2 + a^2\}^2}]\, A_n$$

$$= \frac{4B_1(-)^n\,(2n+1)\pi\cosh(a/2)}{\{(2n+1)^2\pi^2 + a^2\}} +$$

$$+ \frac{2B_2(-)^n\,(2n+1)\,\pi}{\{(2n+1)^2\pi^2 + a^2\}} \times$$

$$\times[\sinh(a/2) - \frac{4a\cosh(a/2)}{\{(2n+1)^2\pi^2 + a^2\}}] +$$

$$+ \frac{B_3(-)^n(2n+1)\,\pi}{2\{(2n+1)^2\pi^2 + a^2\}} \times$$

$$\times\{\cosh(a/2) - \frac{8}{\{(2n+1)^2\pi^2 + a^2\}}\{\cosh(a/2)+a\sinh(a/2$$

$$+ \frac{32\,a^2\cosh(a/2)}{\{(2n+1)^2\pi^2 + a^2\}^2}] \qquad (2.36)$$

Using Eqs.(2.30), (2.32a)-(2.32b) and (2.33a)-(2.33d), the above equation can be written as

$$L_{nn}\,A_n + \sum_{m\neq n} L_{mn}\,A_m = 0\,, \qquad (2.37)$$

where

$$L_{nn} = \gamma_n^2\,\{\frac{1}{\gamma_n^3} - Ra^2\,(1-k-\frac{k}{s})\,\Delta_n(a)\} + \frac{k}{s}\,Ra^2\,(2n+1)^2\pi^2 D_n(a),$$

(2.38a)

$$L_{mn} = \frac{(-)^{m+n}\, 16\pi^2 a\, (2n+1)(2m+1)\gamma_n^2 \gamma_m^2 \cosh^2(a/2).Ra^2(1-k-\frac{k}{s})}{(\sinh a + a)}$$

$$+ (-)^{m+n}\, \frac{k}{s}Ra^2\, (2n+1)(2m+1)\, \pi^2 \gamma_n^2\, D_n(a), \qquad (2.38b)$$

$$\gamma_n = \frac{1}{\{(2n+1)^2\pi^2 + a^2\}}, \qquad (2.38c)$$

$$\Delta_n(a) = 1 - \frac{16\pi^2\, a\, (2n+1)^2\, \gamma_n^2 \cosh^2(a/2)}{(\sinh a + a)}, \qquad (2.38d)$$

and

$$a^3 D_n(a) = 4\gamma_n\, a^2 \coth(a/2) - \coth(a/2) -$$

$$- a + \frac{2a \coth(a/2).\cosh^2(a/2)}{(\sinh a + a)} \qquad (2.38e)$$

Eq. (2.37) is solvable if the determinant vanishes, which forces the value of R, the onset point for stationary convection. Thus R is the root of

$$\text{Det } L_{ij} = 0 \qquad (2.39)$$

If the expansion for Θ in Eq.(2.25) is truncated at the lowest order (i.e., n = 0),

$$\Theta = A_o \cos \pi z$$

and Eq.(2.37) gives

$$L_{oo}\, A_o = 0 . \qquad (2.40)$$

From Eqs.(2.38a)-(2.38e), and Eq.(2.40), the value of $R = (R_s^{(o)})$

in the lowest order approximation is found to be

$$R_s^{(o)} = \frac{(\pi^2 + a^2)^3/a^2}{[(1 - k - \frac{k}{s})G_1(a) + \frac{k}{s} G_2(a)]}, \tag{2.41}$$

where

$$G_1(a) = \Delta_o(a) = 1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2+a^2)^2 (\sinh a + a)} \tag{2.42a}$$

and

$$G_2(a) = -\pi^2 D_o(a) = \frac{\pi^2}{a^2}[1 + \frac{\pi^2-3a^2}{\pi^2+a^2} \frac{\coth(a/2)}{a} - \frac{(1 + \cosh a)}{(\sinh a + a)} \coth(a/2)] \tag{2.42b}$$

The result in Eq.(2.40) is in agreement with one of the variational approximation of Gutkowicz-Krusin et al.[21]

In most of the binary liquids and over a significant range in ^3He-^4He mixture, 'k' is very large compared to s. In this limit ($k/s >> 1$), the value of $R_s^{(o)}$ is minimum for $a \to 0$. The minimum value is given by,

$$-\frac{kR_s^{(o)}}{s} \simeq \lim_{a \to 0} \frac{(\pi^2 + a^2)^3}{a^2[G_1(a) - G_2(a)]} \tag{2.43}$$

As,

$$\lim_{a \to 0} a^2 G_1(a) = \lim_{a \to 0} a^2 \Delta_o(a) \simeq 0 \tag{2.43a}$$

and

$$\lim_{a \to 0} a^2 G_2(a) = -\pi^2 \lim_{a \to 0} a^2 D_o(a) = \frac{2\pi^2 - 24}{3}, \tag{2.44b}$$

Eq. (2.24) gives

$$-\left.\frac{k R_s^{(o)}}{s}\right|_{a \to 0} \simeq \frac{3\pi^6}{24 - 2\pi^2} \simeq 677 . \tag{2.45}$$

The exact numerical value for R is 720. Thus, the maximum error that one encounters in the lowest order approximation is only about 7%. The inclusion of the next order term (n=1) in the expansion of Θ leads to the condition

$$L_{oo} L_{11} - L_{o1} L_{10} = 0 . \tag{2.46}$$

In the limit $k/s >> 1$, according to Eqs.(2.38a) - (2.38b) [as $a \to 0$],

$$L_{oo} L_{11} \simeq \gamma_o^2 \gamma_1^2 [(3 \frac{k}{s} R_s^{(1)})^2 \{\pi^4 a^4 D_o(a) D_1(a)\}$$

$$+ \frac{k}{s} R_s^{(1)} \{ \frac{a^2 \pi^2 D_o(a)}{\gamma_1^3} + \frac{9 a^2 \pi^2 D_1(a)}{\gamma_o^2} \}$$

$$+ \frac{1}{\gamma_o^3 \gamma_1^3}], \tag{2.47a}$$

and

$$L_{o1} L_{10} \simeq \gamma_o^2 \gamma_1^2 (\frac{3k}{s} R_s^{(1)})^2 \pi^4 a^4 D_o(a) D_1(a), \tag{2.47b}$$

because $\lim_{a \to 0} a^2 \Delta_o(a) = 0 = \lim_{a \to 0} a^2 \Delta_1(a)$. (2.47c)

Substituting Eqs.(2.47a) - (2.47b) in Eq. (2.46), the expression for $R_s^{(1)}$ is found as

$$- \frac{k}{s} R_s^{(1)}{}_{a \to 0} = \lim_{a \to 0} \frac{1}{\gamma_o^2 \gamma_1^3 \left[\frac{\pi^2 a^2 D_o(a)}{\gamma_1^3} + \frac{9\pi^2 a^2 D_1(a)}{\gamma_o^3} \right]} \tag{2.48}$$

As $$\lim_{a \to 0} \pi^2 a^2 D_o(a) \simeq \frac{24 - 2\pi^2}{3}$$

and $$\lim_{a \to 0} \pi^2 a^2 D_1(a) \simeq \frac{8 - 6\pi^2}{9},$$

Eq. (2.48) yields

$$- \frac{k}{s} R_s^{(1)}\Big|_{a \to 0} \simeq \frac{729 \pi^6}{5840 - 492 \pi^2} \simeq 712, \tag{2.49}$$

which is within 1% of the exact numerical answer. The results [Eqs. (2.45) and (2.49)] show the existence of stationary convection for zero wave number at the onset. That is, the wavelength is infinite, which means there is only one convection roll in the system at the onset of stationary convection.

2.5 Oscillatory Convection

In this case, the fluctuations are time dependent and they vary periodically. If $\bar{\omega}_o$ be the frequency of oscillation

at the onset then the velocity field and fluctuations in temperature and mass current can be written as

$$W = W_{os}(z) \; e^{i\bar{\omega}_o \tau} , \qquad (2.50a)$$

$$\Theta = \Theta_{os}(z) \; e^{i\bar{\omega}_o \tau} , \qquad (2.50b)$$

$$J = J_{os}(z) \; e^{i\bar{\omega}_o \tau} . \qquad (2.50c)$$

So, for the case oscillatory instability, 'p' is replaced by $i\bar{\omega}_o$ in Eqs.(2.21)-(2.23) and the relevant hydrodynamic equations are:

$$(D^2-a^2)(D^2-b^2)W_{os} = R_{os}a^2(1-k)\Theta_{os} - Ra^2 \cdot k\, J_{os}, \qquad (2.51)$$

$$(D^2-b_1^2)\Theta_{os} = -W_{os}, \qquad (2.52)$$

$$(D^2-b_2^2)\, J_{os} = (D^2-a^2) \; \frac{\Theta_{os}}{s} , \qquad (2.53)$$

where

$$b^2 = a^2 + i\bar{\omega}_o , \qquad (2.54a)$$

$$b_1^2 = a^2 + i\sigma\bar{\omega}_o , \qquad (2.54b)$$

and $$b_2^2 = a^2 + i\, \frac{\sigma}{s}\, \bar{\omega}_o . \qquad (2.54c)$$

The boundary conditions remain the same as in Eqs.(2.24a) - (2.24c). Since Chandrasekhar's procedure yields quite good results for the threshold of stationary convection in the lowest order we begin the study of oscillatory convection by making an ansatz on temperature fluctuation field as

$$\Theta_{os} = A_{os} \cos\pi z \, . \tag{2.55}$$

Inserting the above expression in Eq. (2.53) and then solving it with boundary conditions DJ = 0 at $z = \pm 1/2$, we get the exact solution for the fluctuation in mass current given by

$$J_{os} = \frac{A_{os}}{s} \frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)} \left[\frac{\pi}{b_2} \frac{\cosh(b_2 z)}{\sinh(b_2/2)} + \cos\pi z \right]. \tag{2.56}$$

Now operating by $(D^2-b_2^2)$ on Eq.(2.51) and using Eqs.(2.53) and (2.55), we have

$$(D^2-a^2)(D^2-b^2)(D^2-b_2^2)W_{os} = R_{os}a^2(1-k)(D^2-b_2^2)\Theta_{os} - Ra^2\frac{k}{s}(D^2-a^2)\Theta_{os},$$

$$= -R_{os}a^2[(1-k)(\pi^2+b_2^2)\cos\pi z - \frac{k}{s}(\pi^2 + a^2)] A_{os} \cos\pi z$$

$$= -R_{os} a^2 \tilde{\alpha} A_{os} \cos\pi z \, , \tag{2.57a}$$

where

$$\tilde{\alpha} = [(1-k)(\pi^2 + b_2^2) - \frac{k}{s}(\pi^2 + a^2)]$$

$$= [(1 - k - \frac{k}{s})(\pi^2 + a^2) + (1-k) \frac{i\bar{\omega}_o \sigma}{s}] \, . \tag{2.57b}$$

The differential Eq. (2.56) is to be solved with boundary

conditions given in Eq. (2.24a) and

$$(D^2-a^2)(D^2-b^2)W_{os} = - R_{os}a^2 k \; J_{os} \text{ at } z = \pm 1/2 \tag{2.58}$$

The general solution of Eq. (2.57) is

$$W_{os} = C_1 \cosh(az) + C_2 \cosh(bz) + C_3 \cosh(b_2 z) + \frac{R_{os} \; a^2 \tilde{\alpha} A_{os} \cos \pi z}{(\pi^2 + a^2)(\pi^2+b^2)(\pi^2+b_2^2)} \tag{2.59}$$

The boundary conditions in Eq. (2.58) yields

$$C_3 = A_{os} \; R_{os} \; a^2 \; \frac{k \; \pi}{s \; b_2} \cdot \frac{1}{\sinh(b_2/2)} \cdot \frac{(\pi^2 + a^2)}{(\pi^2+b_2^2)(b_2^2-a^2)(b_2^2-b^2)} \tag{2.60}$$

Other boundary conditions ($W = DW = 0$ at $z = \pm 1/2$) determine other constants, which are found to be

$$C_1 = \frac{1}{\Delta_1}[\{ b_2 \sinh(b_2/2)\cosh(b/2) - b \sinh(b/2)\cosh(b_2/2)\} C_3 - \frac{R_{os} \; a^2 \; \tilde{\alpha} \; A_{os} \cosh(b/2)}{(\pi^2+a^2)(\pi^2+b^2)(\pi^2+b_2^2)}] \tag{2.61a}$$

and

$$C_2 = - [C_1 \; \frac{\cosh(a/2)}{\cosh(b/2)} + C_3 \; \frac{\cosh(b_2/2)}{\cosh(b/2)}], \tag{2.61b}$$

where

$$\Delta_1 = [b \, \mathrm{Sinh}(b/2)\mathrm{Cosh}(a/2) - a \, \mathrm{Sinh}(a/2)\mathrm{Cosh}(b/2)] \qquad (2.61c)$$

Now, inserting Eqs.(2.55) and (2.59) in Eq.(2.52) and multiplying throughout with $\cos\pi z$ in the resulting equation and then integrating from $z = -1/2$ to $z = +1/2$, we find

$$\frac{(\pi^2 + b_1^2)}{2} A_{os} = C_1 \int_{-1/2}^{+1/2} \cosh(az) \cos\pi z \, dz + C_2 \int_{-1/2}^{+1/2} \cosh(bz).\cos\pi z \, dz$$

$$+ C_3 \int_{-1/2}^{+1/2} \cosh(b_2 z) \cos\pi z \, dz + \frac{Ra^2 \tilde{\alpha} A_{os}}{2(\pi^2+a^2)(\pi^2+b^2)(\pi^2+b_2^2)}$$

or,

$$(\pi^2+b_1^2) A_{os} = 4\pi \left[C_1 \frac{\cosh(a/2)}{(\pi^2 + a^2)} + C_2 \frac{\cosh(b/2)}{(\pi^2 + b^2)} + C_3 \frac{\cosh(b_2/2)}{(\pi^2 + b_2^2)}\right]$$

$$+ \frac{R_{os} \, a^2 \tilde{\alpha} A_{os}}{(\pi^2 + a^2)(\pi^2 + b^2)(\pi^2 + b_2^2)} \qquad (2.62)$$

Substituting for C_1, C_2 and C_3 from Eqs.(2.61a), (2.61b) and (2.60) respectively in the above equation and simplifying, we obtain

$$(\pi^2+b_1^2) A_{os} = A_{os} R_{os} a^2 \left[\frac{4\pi^2 k}{sb_2} \cdot \frac{1}{\sinh(b_2/2)} \cdot \frac{1}{(\pi^2+b_2^2)(b_2^2 - a^2)} \times \right.$$

$$\times \left\{ \frac{(b^2-a^2)}{(b_2^2 - b^2)} \times \frac{\Delta_2}{\Delta_1} \cosh(a/2) - \frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)} \cosh(b_2/2) \right\}$$

$$\left. - \frac{4\pi^2 (b^2-a^2) \cosh(a/2) \cos(b/2)}{(\pi^2+a^2)(\pi^2+b^2)^2(\pi^2 + b_2^2)} \cdot \frac{\tilde{\alpha}}{\Delta_1} + \frac{\tilde{\alpha}}{(\pi^2+a^2)(\pi^2+b^2)(\pi^2+b \right. \qquad (2.63)$$

where $\Delta_2 = [b_2 \sinh(b_2/2)\cosh(b/2) - b \sinh(b/2)\cosh(b_2/2)]$

(2.64)

As $A_{os} \neq 0$, the above equation yields

$$R_{os} = \mathrm{Re} \frac{(\pi^2 + b_1^2)/a^2}{[(1 - k - k/s)\tilde{G}_1(a,\bar{\omega}) + \frac{k}{s}\tilde{G}_2(a,\bar{\omega})]}, \qquad (2.65)$$

where

$$\tilde{G}_1(a,\bar{\omega}) = \frac{1}{(\pi^2+a^2)(\pi^2+b^2)} \{1 - \frac{4\pi^2(b^2-a^2)\cosh(a/2)\cosh(b/2)}{\Delta_1} \times$$

$$\times \frac{1}{(\pi^2+a^2)(\pi^2 + b^2)}\} \qquad (2.66a)$$

and

$$\tilde{G}_2(a,\bar{\omega}) = \frac{4\pi^2}{b_2} \cdot \frac{1}{(\pi^2+b_2^2)(b_2^2 - a^2)} [\frac{(\pi^2+ a^2)}{(\pi^2+ b_2^2)} \operatorname{Coth}(b_2/2) -$$

$$- \frac{(b^2-a^2)}{(b_2^2-b^2)} \frac{\Delta_2}{\Delta_1} \frac{\cosh(a/2)}{\sinh(b_2/2)}], \qquad (2.66b)$$

In case $k/s >> 1$, the expression for oscillatory threshold becomes

$$-\frac{k}{s} R_{os} = \mathrm{Re} \frac{(\pi^2 + b_1^2)/a^2}{[\tilde{G}_1(a,\bar{\omega}) - \tilde{G}_2(a,\bar{\omega})]}, \qquad (2.67)$$

where $\tilde{G}_2(a,\bar{\omega})$ is given by Eq.(2.66b) with $k/s >> 1$.

Thus, we find that the technique is equally applicable for both stationary as well as oscillatory instabilities.

REFERENCES

1. Hurle, D.T.J. and Jakeman, E., Phys. Fluids 12, 2704 (1969).

2. Hurle, D.T.J. and Jakeman, E., J. Fluid Mech. 47, 667 (1971).

3. Legros, J.C., Ph.D. Thesis, University of Brussels (1971).

4. Schechter, R.S., Prigogine, I. and Hamm, J.R., Phys. Fluids 15, 379 (1972).

5. Legros, J.C., Platten, J.K. and Poty, P., Phys. Fluids 15, 1383 (1972).

6. Platten, J.K. and Chavepeyer, G., Phys. Fluids 15, 1383 (1972).

7. Verlarde, M.G. and Schechter, R.S., Phys. Fluids 15, 1707 (1972).

8. Legros, J.C., Poty, P. and Thomaes, G., Physica 64, 481 (1973).

9. Schechter, R.S., Velarde, M.G. and Platten, J.K., "Advances in Chemical Physics", 26, ed. by Prigogine, I. and Rice, S.A., 265 (1974).

10. Tokashima, M., J. Phys. Soc. Japan 41, 1394 (1976).

11. Platten, J.K. and Legros, J.C., Phys. Rev. Lett. 45, 1944 (1980).

12. Knobloch, E., Phys. Fluids 23, 1918 (1980).

13. Platten, J.K. and Legros, J.C., "Convection in Liquids", (Springer, NY 1984), Ch. IX.

14. Siggia, E.D., Phys. Rev. B15, 2830 (1977).

15. Meyer, M., Ruppeiner, G. and Ryschkewitsch, M., "Proceedings of the International Conference on Dynamic Critical Phenomena", ed. by Enz, C.P., Springer, NY (1979).

16. Webeler, R.W.H. and Allen, G., Phys. Rev. A5, 1820 (1972).

17. Brand, H., Hohenberg, P.C. and Steinberg, V., Phys. Rev. A27, 591 (1983), also A30, 2548 (1984).

18. Rehberg, I. and Ahlers, G., Phys. Rev. Lett. 55, (1985).

19. Ahlers, G. and Rehberg, I., Phys.Rev.Lett.56,1373 (1986).

20. Gao, H. and Behringer, R., Phys.Rev.A34, 697 (1986).

21. Gutkowicz-Krusin, D., Collins, M.A. and Ross, J. Phys. Fluids 22, 1443 and 1450 (1979).

22. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability", Clarendon Press, Oxford (1961).

23. Landau, L.D. and Lifshitz, E.M., "Fluid Mechanics", Course on Theoretical Physics., 6, Pergamon, NY (1959).

CHAPTER III*

OVERSTABILITY IN MAGNETOHYDRODYNAMIC CONVECTION

3.1 Introduction

The effect of interaction between an external magnetic field and an electrically conducting fluid is two-fold:

i) electric currents are generated because of the fluid motion in the magnetic field and, thus, the applied field is modified;

ii) small fluid parcels carrying current experience an additional force (Lorentz force).

These effects, in general, delay[1] the onset of thermal convection in the system. The possibility of overstability in the magnetohydrodynamic Bénard convection is a problem of long standing. The problem was first analyzed by Chandrasekhar[1]. He used idealized boundary conditions to show that magnetic Prandtl number (p_2) must be greater than thermal Prandtl number (p_1) for overstability to be at all possible. He claimed that the principle of exchange of stability would be valid for $p_2 < p_1$. Thus, he almost ruled out the possibility of overstability. Gibson[2] used realistic boundary conditions and obtained Chandrasekhar's criterion for Q (Chandrasekhar's number) $\gg 1$. Sherman and Ostrach[3] tackled the problem variationally and for the case $Q \gg 1$ found the Chandrasekhar - Gibson answer. Recently,

---

* Contents of this Chapter have been accepted in Phys. Fluids.

Banerjee et al.[4] derived an exact condition, which states that the principle of exchange of stability is valid provided $Q\,p_2 \leq \pi^2$. This gives rise to an interesting possibility that for $p_2 < p_1$, which is true for all practical cases, and for high enough magnetic field, it should be possible to obtain overstability. In this chapter, we investigate this possibility in RB geometry using realistic boundary conditions.

3.2 Magnetohydrodynamic System

A horizontal layer of electrically conducting fluid confined between two rigid and perfectly conducting surfaces with a uniform magnetic field applied across it in vertically upward direction is heated from below. The equation of heat conduction remains the same as in the case of single-component fluid, but the equations of motion (Navier-Stokes equations) change due to the presence of Lorentz force. In Boussinesq approximation, they are

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{V} = -\frac{\vec{\nabla} P}{\rho_m} + \frac{g\,\rho}{\rho_m} + \nu\,\nabla^2\vec{V} + \mu_o(\vec{J} \times \vec{H}), \tag{3.1}$$

where

$$\vec{J} = \sigma(\vec{E} + \mu_o\vec{V} \times \vec{H}) \tag{3.2}$$

is the current density in the fluid, σ is the electrical conductivity and μ_o, the magnetic permeability of the fluid. The electric and magnetic fields are governed by Maxwell's

equations. For magnetohydrodynamic systems, the displacement current is negligible in comparison to the induced current because of fluid motion, and Maxwell's equations are

$$\vec{\nabla} \cdot \vec{H} = 0 , \tag{3.3a}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} , \tag{3.3b}$$

$$\vec{\nabla} \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} , \tag{3.3c}$$

The magnetic field is governed as

$$\frac{\partial \vec{H}}{\partial t} - \vec{\nabla} \times (\vec{V} \times \vec{H}) = \eta_o \nabla^2 \vec{H} \tag{3.4}$$

with $1/\eta_o = (\sigma\mu_o)$.

If h, θ be the induced magnetic field, fluctuation in temperature field respectively at the onset of convection, then the z-component of Eq.(3.1) after elimination of $\vec{\nabla}p$ from it and the equation for induced field are

$$\nabla^2\left(\frac{\partial}{\partial t} - \nabla^2\right)w = -g\alpha\left(\nabla^2\theta - \frac{\partial^2\theta}{\partial z^2}\right) + \frac{\mu}{\rho_m} H_s \frac{\partial}{\partial z}(\nabla^2 h) \tag{3.5}$$

and

$$\frac{\partial h}{\partial t} = \eta_o \nabla^2 h + H_s \frac{\partial w}{\partial z} , \tag{3.6}$$

where H_s is the magnetic field distribution in conduction state. Now writing w and h as

$$w = w^*(z,t) \; e^{i(k_x x + k_y y)} , \tag{3.7a}$$

$$h = h^*(z,t) \; e^{i(k_x x + k_y y)} , \tag{3.7b}$$

where $\bar{k} = \sqrt{k_x^2 + k_y^2}$ is a two-dimensional wave number in XY plane. Now inserting Eqs.(3.7a)-(3.7b) in Eqs.(3.5)-(3.6) and non-dimensionalizing the resulting equations in linearized from, we obtain

$$(D^2-a^2)(D^2-a^2 - \frac{\partial}{\partial\tau})w^*(z,t) = Ra^2\theta + Q(D^2-a^2)D\,h^*(z,t)$$

and (3.8)

$$(D^2-a^2 - p_2\frac{\partial}{\partial\tau})h^*(z,t) = Dw^*(z,t)$$

where

$$Q = \frac{\mu_o H_s^2 d^2}{\rho_m \nu\eta_o}$$ is Chandrasekhar's number,

R, the Rayleigh number, d, the separation between the bounding surfaces and p_2, the magnetic Prandtl number.

Thus the relevant hydrodynamic equations for the simple magnetohydrodynamic Benard problem are

$$(D^2-a^2)(D^2-a^2 - \frac{\partial}{\partial\tau})w^* = Ra^2 + Q(D^2-a^2)Dh^*, \tag{3.9}$$

$$(D^2 - a^2 - p_1\frac{\partial}{\partial\tau})\theta = -w^*, \tag{3.10}$$

$$(D^2 - a^2 - p_2\frac{\partial}{\partial\tau})h^* = Dw^*. \tag{3.11}$$

3.3 Boundary Conditions

The boundary surfaces considered here are assumed to be perfectly conducting. This means

$$h^* = 0 \text{ at } z = \pm 1/2 .$$

Other boundary conditions because of rigidity of the bounding

surfaces are

$$w^* = Dw^* = \theta = 0 \text{ at } z = \pm 1/2 .$$

3.4 Mathematical Analysis

Since the analysis is intended to look for any possibility of overstability, the variables are expressed as

$$\begin{pmatrix} w^* \\ \theta \\ h^* \end{pmatrix} = \begin{pmatrix} W(z) \\ \Theta(z) \\ K(z) \end{pmatrix} e^{i\omega_o\tau} , \qquad (3.12)$$

where 'ω_o' is the frequency of the convection roll at the onset of overstability. Therefore, Eqs.(3.9)-(3.11) become

$$(D^2-a^2)(D^2-b^2)W = Ra^2\Theta + Q(D^2-a^2)DK, \qquad (3.13)$$

$$(D^2 - \beta^2)\ \Theta = -W, \qquad (3.14)$$

$$(D^2 - \gamma^2)\ K = DW, \qquad (3.15)$$

where

$$b^2 = a^2 + i\omega_o,\ \beta^2 = a^2 + i\omega_o p_1 \text{ and } \gamma^2 = a^2 + i\omega_o p_2 . \qquad (3.16)$$

The boundary conditions are now

$$W = DW = \Theta = K = 0 \text{ at } z = \pm 1/2 . \qquad (3.17)$$

Following the procedure discussed earlier, the temperature field is expanded in Fourier cosine series as

$$\Theta = \sum_{n=0}^{\infty} A_n \cos(2n+1)\pi z . \tag{3.18}$$

Now Eq.(3.14) is inserted in Eq.(3.15), which is then solved using Eq.(3.18) and boundary conditions on K . The solution is given by

$$K = \sum_n (2n+1)A_n \frac{(2n+1)^2\pi^2+\beta^2}{(2n+1)^2\pi^2+\gamma^2}\left[\sin(2n+1)\pi z - (-)^n \times \frac{\sinh(\gamma z)}{\sinh(\gamma/2)}\right]. \tag{3.19}$$

Inserting the expansion for Θ and K in Eq.(3.13), the latter becomes

$$[(D^2-a^2)(D^2-b^2) - Q D^2]W = \sum_n A_n \left[Ra^2 + i\omega_o p_2 Q(2n+1)^2\pi^2 \times \frac{(2n+1)^2\pi^2+\beta^2}{(2n+1)^2\pi^2+\gamma^2}\right]\cos(2n+1)\pi z -$$

$$- i\omega_o p_2 \gamma Q \sum_n A_n (2n+1)\pi(-)^n \frac{(2n+1)^2\pi^2+\beta^2}{(2n+1)^2\pi^2+\gamma^2} \times \frac{\cosh(\gamma z)}{\cosh(\gamma/2)} . \tag{3.20}$$

The general solution of this equation is

$$W = \sum_n A_n \frac{P_{1n}}{\{(2n+1)^2\pi^2+q_1^2\}\{(2n+1)^2\pi^2+q_2^2\}\{(2n+1)^2\pi^2+\gamma^2\}}\cos(2n+1)\pi z$$

$$+ \sum_n A_n P_{2n}\cosh(\gamma z) + B_1\cosh(q_1 z) + B_2\cosh(q_2 z), \tag{3.21}$$

where

$$P_{1n} = [Ra^2\{(2n+1)^2\pi^2+\gamma^2\} + i\omega_o p_2 Q(2n+1)^2\pi^2\{(2n+1)^2\pi^2+\beta^2\}] \tag{3.22}$$

$$P_{2n} = \frac{i\omega p_2 \gamma Q(2n+1)\pi(-)^n\{(2n+1)^2\pi^2 + \beta^2\}}{\{(2n+1)^2\pi^2 + \gamma^2\}\{Q\gamma^2 - (\gamma^2-a^2)(\gamma^2-b^2)\} \sinh(\gamma/2)} \tag{3.23}$$

and

$$q_{1,2} = \frac{1}{2}\left[\{Q + (a+b)^2\}^{1/2} \pm \{Q + (a-b)^2\}^{1/2}\right]. \tag{3.24}$$

The boundary conditions on W give

$$B_1 \cosh(q_1/2) + B_2 \cosh(q_2/2) = -\sum_n A_n P_{2n} \cosh(\gamma/2) \tag{3.25}$$

and

$$q_1 B_1 \sinh(q_1/2) + q_2 B_2 \sinh(q_2/2) = -\sum_n P_{2n}\gamma \sinh(\gamma/2) + \sum_n A_n \tilde{P}_{1n}(2n+1)(-)^n\pi \tag{3.26}$$

leading to

$$B_1 = \frac{\left[-q_2 \sinh(\frac{q_2}{2}) \sum_n A_n P_{2n} \cosh(\gamma/2) + \sum_n A_n P_{2n}\gamma \sinh(\gamma/2) \cosh(\frac{q_2}{2}) - \cosh(q_2/2)\sum_n A_n \tilde{P}_{1n}(2n+1)(-)^n\pi\right]}{\left[q_2 \sinh(\frac{q_2}{2})\cosh(\frac{q_1}{2}) - q_1 \sinh(\frac{q_1}{2})\cosh(\frac{q_2}{2})\right]} \tag{3.27}$$

106284

and

$$B_2 = -\left[\sum_n A_n P_{2n} \cosh(\gamma/2) + B_1 \cosh(q_1/2)\right] / \cosh(q_2/2) \tag{3.28}$$

with

$$\tilde{P}_{1n} = P_{1n}/[\{(2n+1)^2\pi^2 + q_1^2\}\{(2n+1)^2\pi^2 + q_2^2\}\{(2n+1)^2\pi^2 + \gamma^2\}]. \tag{3.29}$$

Inserting the solution given in Eq.(3.21) together with the expansion of Eq.(3.18), in Eq.(3.14), multiplying both sides with $\cos(2n+1)\pi z$ and integrating from $z = -1/2$ to $z = +1/2$, the following expression is obtained

$$\{(2n+1)^2\pi^2 + q_1^2\}\{(2m+1)^2\pi^2 + q_2^2\}\{(2m+1)^2\pi^2 + \beta^2\}\{(2m+1)\pi^2 + \gamma^2\}A_m$$

$$= A_m P_{1m}[1 - \Delta_{mm}] - \sum_{n \neq m} A_n P_{1n} \Delta_{mn} +$$

$$+ \sum_n A_n P_{2n} L_m [(2m+1)^2\pi^2 + \gamma^2], \qquad (3.30)$$

where

$$\Delta_{mn} = \frac{4\pi^2(2n+1)(2m+1)(-)^{m+n}(q_1^2 - q_2^2)\cosh(\frac{q_1}{2})\cosh(\frac{q_2}{2})}{\{(2m+1)^2\pi^2 + q_1^2\}\{(2m+1)^2\pi^2 + q_2^2\}[q_1 \sinh(\frac{q_1}{2})\cosh(\frac{q_2}{2}) - q_2 \sinh(\frac{q_2}{2})\sinh(\frac{q_1}{2})]} \qquad (3.31)$$

and

$$L_m = (-)^m 4(2m+1)\pi\left[\frac{(q_2^2 - \gamma^2)\{(2n+1)^2\pi^2 + q_1^2\}\cosh(\gamma/2)}{\{(2m+1)^2\pi^2 + \gamma^2\}} + \right.$$

$$\left. + \frac{(q_1^2 - q_2^2)\cosh(\frac{q_1}{2})\{s\sinh\frac{\gamma}{2}\cosh(\frac{q_2}{2}) - q_2\cosh(\frac{\gamma}{2})\sinh(\frac{q_2}{2})\}}{\{q_1\sinh(\frac{q_1}{2})\cosh(\frac{q_2}{2}) - q_2\sinh(\frac{q_2}{2})\cosh(\frac{q_1}{2})\}}\right]. \qquad (3.32)$$

Defining,

$$G_{mm} = [\{(2m+1)^2\pi^2 + q_1^2\}\{(2m+1)^2\pi^2 + q_2^2\}\{(2m+1)^2\pi^2 + \beta^2\}$$

$$\{(2m+1)^2\pi^2 + \gamma^2\}] - P_{1m}(1 - \Delta_{mm}) - P_{2m}L_m\{(2m+1)^2\pi^2 + \gamma^2\}$$

(3.33)

and

$$G_{mn} = P_{1n}\Delta_{mn} - P_{2n}L_m\{(2m+1)^2\pi^2 + \gamma^2\}, \qquad (3.34)$$

Eq. (3.30) can be rewritten as

$$G_{mm}A_m + \sum_{n \neq m} G_{mn}A_n = 0 \;. \qquad (3.35)$$

The consistency condition is

$$\text{Det } G = 0. \qquad (3.36)$$

The entries of the determinant being complex numbers, yield two equations which are solved for R and ω_o.

In principle this yields the exact solution. In practice, the evaluation of large determinants is not necessary. As shown in Chapters I and II, the lowest order truncation yields quite accurate results. In fact one-mode ($n = 0$) truncation is extremely good for a single-component fluid.

3.5 Overstability

For overstability one-mode truncation should yield accurate results in the low frequency regime. At high frequencies the truncation is susceptible to error as larger

determinants are going to produce higher powers of ω_o and lead to inconsistency. Now the analysis can be simplified by considering the limit $Q >> 1$, $p_2 << 1$ with $Qp_2 (> \pi^2)$ finite. Equating real and imaginary parts of Eq.(3.36) in its lowest order approximation, one obtains

$$(\pi^2+a^2) p_1 p_2 \omega_o^2 \Delta = (\pi^2+a^2)^3 (1+p_1+p_2) + Q\pi^2(\pi^2+a^2) \times$$
$$\times (p_1+p_2\Delta - p_2) - Ra^2 p_2 \qquad (3.37)$$

and

$$Ra^2 = (\pi^2+a^2)^3 + Q\pi^2(\pi^2+a^2) - \frac{\omega_o^2}{(\pi^2+a^2)} \{Q\pi^2 p_1 p_2(\Delta - 1)$$
$$+ (\pi^2+a^2)^2 (p_1+p_2\Delta + p_1 p_2\Delta)\}, \qquad (3.38)$$

where

$$\Delta = 1 + \frac{4a \coth(a/2)}{(\pi^2 + a^2)} . \qquad (3.39)$$

For idealized boundary conditions $\Delta = 1$ and the equations become similar to those obtained by Chandrasekhar. It is the use of proper boundary conditions, which make $\Delta \neq 1$ and lead to the possibility of overstability even though $p_2 << p_1$. Eliminating R from Eq.(3.37) by using Eq.(3.38), and working to the lowest order in p_2 yields

$$\Omega^2 = \frac{p_2}{p_1} \left[\frac{Q\pi^2 p_1}{(\Delta - 1)} + \frac{(\pi^2 + a^2)^2(1+p_1)}{(\Delta - 1)}\right], \qquad (3.40)$$

where $\Omega = \omega_o p_2$.

Since the wave number 'a' increases with 'Q' and $\Delta \rightarrow 1$ for $Q >> 1$, the Q dependence of various quantities are self consistent if, in accordance with the case for stationary instability, $a \sim Q^{1/6}$. In this case, the first term of Eq. (3.40) dominates and

$$\Omega^2 \sim Q\, p_2\, \pi^2/(\Delta-1) \sim Q\, p_2\, \pi^2(a/4) \quad . \tag{3.41}$$

It is interesting to note that Qp_2 enters as a combination in the expression for the frequency of overstability. The Rayleigh number at this order of calculation is

$$Ra^2 \simeq (\pi^2+a^2)^3 + Q\,\pi^2(\pi^2+a^2) - \frac{p_1}{p_2^2}\,\frac{(Q\pi^2 p_2)^2}{(\pi^2+a^2)^2}\,, \tag{3.42}$$

which is lower than the threshold R_s for stationary convection given by

$$R_s\, a^2 = (\pi^2 + a^2)^3 + Q\pi^2(\pi^2+a^2). \tag{3.43}$$

Thus the possibility of overstability even if $p_2 << p_1$ is not ruled out.

3.6 Necessary Condition for Existence of a Polycritical Point

If there is an overstable convection, it is relevant to ask whether there is a point in parameter space where the lines of oscillatory and stationary convections meet. This will indicate the possibility of any polycritical point where

both types of phenomena are possible in the system. The condition for this possibility is obtained by equating the expression for ω_o^2 to zero. With one-mode truncation and in the limit $Q >> 1$, this condition [from Eqs (3.37)-(3.39)] becomes

$$Q\,\pi^2\,(\pi^2+a^2)(p_1-p_2) + (1+p_1)(\pi^2+a^2)^3 + 4\pi p_2\,Q\,a\coth(a/2) = 0 \qquad (3.44)$$

Since the last two terms on the l.h.s. are positive definite, the condition is fulfilled only if

$$p_1 < p_2 ,$$

which is a necessary condition for the existence of codimension-2 bifurcations[5-6]. Thus Chandrasekhar-Gibson criterion is seen to be a necessary condition for the existence of a polycritical point and not a necessary condition for overstability.

REFERENCES

1. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability",[Clarendon Press, Oxford (1961)], Chap.4.

2. Gibson, R.D., Proc. Cambridge Philos. Soc. 62, 287 (1966).

3. Shermen, M. and Ostrach, S., J. Fluid Mech. 24, 661 (1966).

4. Banerjee, M.B., Gupta, J.R., Shandil, R.G., Sood, S.K., Banerjee, B. and Banerjee, K., J. Math. Anal. Appl. 108, 216 (1985).

5. Brand, H., Hohenberg, P.C. and Steinberg, V., Phys. Rev. A27, 591 (1983); 30, 2548 (1984).

6. Rehberg, I. and Ahlers, G., Phys. Rev. Lett. 55, 500 (1985).

CHAPTER IV*

CONVECTION IN MODULATED RAYLEIGH-BÉNARD FLOW

4.1 Introduction

In this chapter the onset of convective instability in a modulated Rayleigh-Bénard system with realistic boundaries is discussed. An external modulation of the temperature difference between the plates can drastically modify the behaviour of the system. The study of the effect of modulation is currently receiving considerable attention, spurred by highly sophisticated experimental techniques. A proper tuning of amplitude and frequency of modulation can lead to a new class of phenomena and thus makes the problem of hydrodynamic instability more versatile and interesting.

The theoretical treatment[1-10] of the effect of modulation is based either on full hydrodynamic equations[1-4] or a few-mode truncation thereof[5-10]. While the few-mode truncation always yields solutions in closed form, the hydrodynamic equations seem to yield closed-form solutions only with idealized (free) boundary conditions. Here, using Chandrasekhar's procedure, the critical Rayleigh number is obtained in closed form for the system with proper rigid boundary conditions.

---

* Contents of this chapter have been accepted in Phys.Rev. A.

4.2 Hydrodynamics of Modulated Rayleigh-Bénard Flow

The control parameter — the temperature difference between the two plates — is modulated by imposing an external time-periodic disturbance on the temperature of the lower plate. The temperature (T_1) of the lower plate is

$$T_1 = T_1 + \varepsilon \, Re(\Delta T) \, e^{i\omega t}, \qquad (4.1)$$

where ε and ω are the amplitude and frequency of the modulation respectively, T_1 and T_2 are the temperatures of lower and upper plates respectively in the absence of any modulation, $\Delta T = T_1 - T_2$. Since the equation governing the conduction of heat is linear and is given by

$$\frac{\partial T_H}{\partial t} = \lambda \frac{\partial^2 T_H}{\partial Z^2} , \qquad (4.2)$$

the temperature field (T_H) of the hydrodynamic system can be expressed as

$$T_H(Z,t) = T_s(Z) + \varepsilon \, T_{os}(Z,t) \; . \qquad (4.3)$$

'T_s' is the steady state temperature field in the absence ofany modulation, and is given as

$$T_s(Z) = T_1 - (\Delta T)(\frac{Z}{d} + \frac{1}{2}) \; . \qquad (4.4)$$

If 'Z' is measured in terms of the separation 'd' between the plates, the above equation takes the form

$$T_s(z) = T_1 - (\Delta T)(z + \frac{1}{2}), \qquad (4.5a)$$

$$T_H = T_1 - \Delta T(z + \tfrac{1}{2}) + \varepsilon Re(\Delta T) \frac{\sinh\{\alpha d(\frac{1}{2} - z)\}}{\sinh(\alpha d)} e^{i\omega t} \quad (4.10)$$

If θ, as defined earlier (Chapter I), be the fluctuation in the temperature field at the onset of convection, the modified temperature field will be

$$T = T_H + \theta . \quad (4.11)$$

Now the linearized perturbation equations, as obtained in Chapter I, are

$$(D^2-a^2)(D^2 - a^2 - \frac{\partial}{\partial\tau}) w = \theta \quad (4.12)$$

and

$$(D^2-a^2 - p_1 \frac{\partial}{\partial\tau}) \theta = - Ra^2 [1 + Re f(z) e^{i\bar{\omega}\tau}] w ,$$

$$(4.13)$$

where

$$f(z) = \frac{\alpha d}{\sinh(\alpha d)} \cosh\{\alpha d(\tfrac{1}{2} - z)\}, \quad (4.14)$$

and θ, $\bar{\omega}$ and τ are measured in units of ΔT, $\frac{\nu}{d^2}$ and $\frac{d^2}{\nu}$ respectively. p_1 and R have their usual meaning.† As the realistic bounding surfaces are rigid, the boundary conditions are

$$w = Dw = \theta = 0 \text{ at } z = \pm \tfrac{1}{2} . \quad (4.15)$$

4.3 Mathematical Analysis

The critical value R can be obtained by solving the Eqs.(4.12)-(4.13) with boundary conditions given by Eq.(4.15). For small modulation ($\varepsilon \ll 1$), the solutions are

---

† p is thermal Prandtle number and R, the Rayleigh number

obtained by expanding w, θ and R in powers of 'ε'. That is,

$$w = w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots , \tag{4.16}$$

$$\theta = \theta_o + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots , \tag{4.17}$$

$$R = R_o + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots . \tag{4.18}$$

Inserting these expansions in Eqs.(4.12)-(4.13) and (4.15) and equating terms of the same order in 'ε', the resulting equations up to $O(\varepsilon^2)$ are

$$(D^2-a^2)(D^2-a^2 - \frac{\partial}{\partial\tau})\, w_o = \theta_o, \tag{4.19a}$$

$$(D^2-a^2 - p_1\frac{\partial}{\partial\tau})\, \theta_o = - R_o a^2 w_o; \tag{4.19b}$$

$$(D^2-a^2)\,(D^2 - a^2 - \frac{\partial}{\partial\tau}) w_1 = \theta_1, \tag{4.20a}$$

$$(D^2-a^2 - p_1\frac{\partial}{\partial\tau})\theta_1 = - R_o a^2 w_1 - R_1 a^2 w_o - R_o a^2 \,\mathrm{Re}(f w_o e^{i\bar{\omega}\tau}); \tag{4.20b}$$

and

$$(D^2-a^2)(D^2 - a^2 - \frac{\partial}{\partial\tau}) w_2 = \theta_2, \tag{4.21a}$$

$$(D^2-a^2-p_1\frac{\partial}{\partial\tau})\theta_2 = -R_o a^2 w_2 - R_2 a^2 w_o - R_o a^2 \,\mathrm{Re}(f w_1 e^{i\bar{\omega}\tau}) - R_1 a^2 \,\mathrm{Re}(f w_o e^{i\bar{\omega}\tau}) \tag{4.21b}$$

The zeroth order equations represent the system in the absence of any modulation. In this situation the principle of exchange of stabilities is valid and these equations are reduced to

$$(D^2 - a^2)^2 w_o = \theta_o \tag{4.22a}$$

and
$$(D^2 - a^2)\theta_o = -R_o a^2 w_o \quad . \tag{4.22b}$$

The realistic boundary conditions are

$$\theta_o = w_o = Dw_o = 0 \quad \text{at} \quad z = \pm 1/2 \ . \tag{4.23}$$

The solutions will be symmetric at the lowest Rayleigh number. Chandrasekhar's[11] technique is applied to solve these equations. As one-mode truncation yields extremely accurate results (Chapter I) for a single-component fluid, the temperature fluctuation 'θ' is expanded in Fourier cosine series and truncated at the first term. Therefore,

$$\theta_o = A_o \cos\pi z \ . \tag{4.24}$$

Inserting the above in Eq.(4.22a), the solution for w_o with boundary conditions on w_o [Eq. (4.23)] becomes

$$w_o = B_o \cosh(az) + C_o z \sinh(az) + \frac{A_o \cos\pi z}{(\pi^2 + a^2)^2} \ , \tag{4.25}$$

where

$$2B_o = - C_o \tanh (a/2) \tag{4.26}$$

and

$$C_o = \frac{2\pi A_o \cosh(a/2)}{(\pi^2+a^2)^2(\sinh + a)} \ . \tag{4.27}$$

Inserting Eqs.(4.24) and (4.25) in Eq.(4.22b),

multiplying throughout with $\cos\pi z$ and integrating from $z = -1/2$ to $z = 1/2$ yields

$$R_o = \frac{(\pi^2+a^2)^3}{a^2}\left[1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2+a^2)^2(\sinh a + a)}\right]^{-1}, \tag{4.28}$$

which is the same result as obtained in Chapter I[Eq.(1.33)].

The perturbative corrections R_1, R_2 etc., which are compatible with R_o may be obtained by requiring that Eqs.(4.20a)-(4.20b) and Eqs.(4.21a)-(4.21b) must have solutions compatible with the solutions of Eqs.(2.19a)-(2.19b). The solvability criterion for Eqs.(4.21a)-(4.21b) demands that

$$\frac{1}{T}\int_0^T (\tilde{w}_o\ \tilde{\theta}_o)\begin{pmatrix} 0 \\ -R_o a^2 w_o - R_o a^2 \mathrm{Re}(f w_o e^{i\bar{\omega}\tau}) \end{pmatrix} d\tau = 0, \tag{4.29}$$

where $\tilde{w}_o$ and $\tilde{\theta}_o$ are left reactors of Eqs.(2.19a)-(2.19b). It leads to

$$R_1 = 0 . \tag{4.30}$$

To obtain the forms of w_1 and θ_1 within the one mode Fourier approximation described above, fw_o is expanded in Fourier series and only the $\cos\pi z$ term is retained. We write

$$fw_o \simeq F\cos\pi z , \tag{4.31}$$

where

$$F = 2 \int_{-1/2}^{+1/2} f\, w_o \cos \pi z \, dz$$

$$= 2 \left[\frac{A_o}{(\pi^2+a^2)^2} \cdot \frac{2\pi^2}{\{4\pi^2+(\alpha d)^2\}} + \frac{B_o\, \alpha d}{\sinh(\alpha d)} \pi \cosh\left(\frac{\alpha d}{2}\right) \times \right.$$

$$\times \left[\frac{\cosh\{\frac{1}{2}(a-\alpha d)\}}{\{(a-\alpha d)^2+\pi^2\}} + \frac{\cosh\{\frac{1}{2}(a+\alpha d)\}}{\{(a+\alpha d)^2+\pi^2\}} \right] + \frac{C_o\, \alpha d}{\sinh(\alpha d)} \times$$

$$\times \pi \cosh\left(\frac{\alpha d}{2}\right) \left[\frac{\sinh\{\frac{1}{2}(a-\alpha d)\}}{\{(a-\alpha d)^2+\pi^2\}} + \frac{\sinh\{\frac{1}{2}(a+\alpha d)\}}{\{(a+\alpha d)^2+\pi^2\}} \right] -$$

$$- \frac{4(a-\alpha d)}{\{(a-\alpha d)^2+\pi^2\}^2} \cosh\left\{\frac{(a-\alpha d)}{2}\right\} - \frac{4(a+\alpha d)}{\{(a+\alpha d)^2+\pi^2\}^2} \times$$

$$\times \cosh\left\{\frac{(a+\alpha d)}{2}\right\} \quad . \tag{4.32}$$

The time dependence of w_1 and θ_1 is of the form $e^{i\bar{\omega}\tau}$, so that w_1 and θ_1 can be written as

$$\theta_1 = \Theta_1(z)\, e^{i\bar{\omega}\tau} , \tag{4.33a}$$

$$w_1 = W_1(z)\, e^{i\bar{\omega}t} . \tag{4.33b}$$

Substituting above in Eqs.(4.20a)-(4.20b) with $R_1 = 0$, we find

$$(D^2-a^2)(D^2-b^2)\, W_1 = \Theta_1 , \tag{4.34a}$$

$$(D^2-b_1^2)\,\Theta_1 = -R_o a^2 W_1 - R_o a^2 f w_o\, e^{2i\bar{\omega}\tau} , \tag{4.34b}$$

where,

$$b^2 = a^2 + i\,\bar{\omega} \tag{4.35a}$$

and

$$b_2^2 = a^2 + i\bar{\omega}\; p_1 . \tag{4.35b}$$

The boundary conditions are

$$\Theta_1 = W_1 = DW_1 = 0 \text{ at } z = \pm 1/2 . \qquad (4.36)$$

By proceeding, as for zeroth order solution, by making the ansatz

$$\Theta_1 = A_1 \cos\pi z , \qquad (4.37)$$

the solution for W_1 is obtained from Eq.(4.34a) with boundary conditions on W_1 in Eq.(4.36). The solution is found to be

$$W_1 = B_1 \cosh(az) + C_1 \cosh(bz) + \frac{A_1 \cos\pi z}{(\pi^2+a^2)(\pi^2+b^2)} , \qquad (4.38)$$

with

$$B_1 \cosh(a/2) = -C_1 \cosh(b/2) \qquad (4.39)$$

and

$$C_1 = A_1 \frac{\cosh(a/2)}{[b \sinh(b/2)\cosh(a/2) - a \sinh(a/2)\cosh(b/2)]} \cdot \frac{1}{(\pi^2+a^2)} \cdot \frac{1}{(\pi^2+b^2)} . \qquad (4.40)$$

The amplitude A_1 is obtained from Eq.(4.20b) by using the above equation [Eq.(4.37)] and Eqs.(4.38)-(4.40) as

$$A_1 = \frac{R_o a^2 F}{(\pi^2+a^2+p_1 i\bar{\omega}) - \frac{R_o a^2}{(\pi^2+a^2)(\pi^2+b^2)}\left[1 + \frac{4\pi^2(a^2-b^2)\cosh(a/2)\cosh(b/2)}{(\pi^2+a^2)(\pi^2+b^2)\{b \sinh(b/2) \cdot \cosh(a/2) - a \sinh(a/2) \cdot \cosh(b/2)\}}\right]} \qquad (4.41)$$

Now turning to $O(\varepsilon^2)$ equations and applying solvability condition, we find that

$$\frac{1}{T}\int_0^T [R_2 a^2 < \tilde{\theta}_o \mid w_o > + R_o a^2 < \tilde{\theta}_o \mid f w_1 \, e^{i\bar{\omega}\tau} >] d\tau = 0 \ . \tag{4.42}$$

Inserting the expressions for w_1 and taking the time average, we arrive at

$$\begin{aligned} R_2 &= -R_o \ \mathrm{Re} \ \frac{< \tilde{\theta}_o \mid f W_1 >}{2 < \tilde{\theta}_o \mid w_o >} \\ &= -R_o \ \mathrm{Re} \ \frac{< (\pi^2 + a^2)\theta_o \mid f W_1 >}{2 < (\pi^2 + a^2)\theta_o \mid w_o >} \\ &= -R_o \ \mathrm{Re} \ \frac{< (D^2 - a^2)\theta_o \mid f \ W_1 >}{2 < (D^2 - a^2) \ \theta_o \mid w_o >} \\ &= -R_o \ \mathrm{Re} \ \frac{< w_o \mid f \ W_1 >}{2 < w_o \mid w_o >} \\ &= -R_o \ \mathrm{Re} \ \frac{< W_1 \mid f \ w_o >}{< w_o \mid w_o >} \ . \end{aligned} \tag{4.43}$$

Evaluating $$< W_1 \mid f w_o > = \int_{-1/2}^{+1/2} W_1^* \ f \ w_o \ dz \tag{4.44a}$$

and $$< w_o \mid w_o > = \int_{-1/2}^{+1/2} w_o^* \ w_o \ dz \tag{4.44b}$$

and inserting in Eq.(4.43), we find

$$R_2 = \frac{R_o^2 \ a^2 \ |F|^2}{4 < w_o \mid w_o >} \ \mathrm{Re} \ \frac{G(\bar{\omega})}{R_o a^2 G(\bar{\omega}) - (\pi^2 + a^2)(\pi^2 + a^2 + i\bar{\omega})(\pi^2 + a^2 + i p_1 \bar{\omega})} \ , \tag{4.45}$$

where

$$G(\bar{\omega}) = 1 + \frac{4\pi^2(a^2 - b^2)\cosh(a/2)\cosh(b/2)}{(\pi^2+a^2)(\pi^2+b^2)[b\sinh(b/2)\cosh(a/2)-a\sinh(a/2)\cosh(b/2)]} \tag{4.46}$$

4.4 Results and Discussions

It is relevant to look at the asymptotic behaviours of R_2/R_o —— the fractional correction to the critical Rayleigh number because of external modulation. In zero frequency limit, $G(\bar{\omega})$ is expanded in powers of $i\bar{\omega}$ as

$$G(\bar{\omega}) = G(o) + (i\bar{\omega})\, G'(\bar{\omega})\Big|_{\bar{\omega}=0} + (i\bar{\omega})^2\, G''(\bar{\omega})\Big|_{\bar{\omega}=0} + \ldots\ldots, \tag{4.47}$$

where

$$G(o) = 1 - \frac{16\pi^2 a\cosh^2(a/2)}{(\pi^2+a^2)^2(\sinh a+a)} \tag{4.48a}$$

and

$$G'(\bar{\omega})\Big|_{\bar{\omega}=o} << \frac{(\pi^2+a^2)^2(p_1+1)}{R_o\, a^2} \tag{4.48b}$$

and the second derivative contributes even less.

In this limit, the fractional correction turns out to be

$$\frac{R_2(\bar{\omega}=o)}{R_o} \simeq \frac{1}{2}\left[\frac{p_1}{(p_1+1)^2}\right], \tag{4.49}$$

which is same as that with idealized boundary conditions[1].

That is for very small frequencies, the modulation stabilizes the conduction state and correction to the critical Rayleigh number is independent of the types of bounding surfaces considered.

In very high frequency ($\bar{\omega} \to \infty$) regime,

$$G(\bar{\omega}) \to 1, \quad |F(\bar{\omega})|^2 \sim \frac{1}{\bar{\omega}^3}$$

and thus

$$\frac{R_2}{R_o} \simeq \frac{1}{\bar{\omega}^5} \quad . \tag{4.50}$$

The correction is positive definite and monotonic. It is different than the results for idealized boundary conditions, where the fractional correction is proportional to $\bar{\omega}^{-6}$.

Now we compare our results with the relevant numerical work by Rosenblat and Tanaka[12]. Choosing the amplitude of modulation $\varepsilon = 0.4$, our results agree with that of the above mentioned numerical work for both $p_1 = 10$ and $p_1 = 1$ within 5% for all frequencies. This establishes the soundness of the procedure applied in the present work.

REFERENCES

1. Venezian, G., J. Fluid Mech. 35, 243 (1969).

2. Rosenblat, S. and Herbert, D.M., J. Fluid Mech. 43, 385 (1970).

3. Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. B30, 1336 (1984).

4. Banerjee, K. and Bhattacharjee, J.K., Proc. R. Soc. London A395, 353 (1984).

5. Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. Lett. 51, 2284 (1983).

6. Ahlers, G. and Hohenberg, P.C., Phys. Rev. Lett. 53, 48 (1985).

7. Agarwal, A., Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. B30, 6458 (1984).

8. Ahlers, G. and Hohenberg, P.C., Phys. Rev. A32, 3493 (1984).

9. Zielinska, B.J.A., Mukamel, D., Steinberg, V. and Fishman, S., Phys. Rev. A32, 702 (1985).

10. Agarwal, A., Bhattacharjee, J.K. and Banerjee, K., Phys. Lett. 111A , 379 (1985).

11. Chandrasekhar, S.,"Hydrodynamic and Hydromagnetic Stability"(Clarendon Press, Oxford 1961).

12. Rosenblat, S. and Tanaka, G.A., Phys. Fluids 14, 1349 (1971).

CHAPTER V *

CONVECTION IN MODULATED BINARY LIQUIDS

5.1 Introduction

RB convection in binary liquids[1-3] has of late attracted a good deal of attention. The possibility of oscillatory convection[4,5] at the onset of hydrodynamic instability and subsequent codimension-two bifurcations[6] has prompted the recent spate of experiments. Because of the above two possibilities, parametric modulation of the temperature difference between the two plates is expected to produce interesting effects on the flow pattern. In this chapter, the effect of modulation on the onset of stationary as well as oscillatory convection is investigated for binary liquid mixture confined to realistic bounding surfaces. Chandrasekhar's procedure[7] is applied to incorporate the realistic boundary conditions.

5.2 Hydrodynamics of Modulated Binary Liquids

Equations describing the fluid motion, temperature and concentration fields remain similar to those given in Chapter II[Eqs.(2.1)-(2.3)] . The incompressibility condition and the equation of state are also given by Eqs.(2.4)-(2.6). As the temperature of the lower plate is modulated as

$$T_1 = T_1 + \varepsilon \operatorname{Re} (\Delta T)\, e^{i\omega t}, \tag{5.1}$$

* Contents of this chapter have been submitted for publication.

the temperature and concentration profiles in steady state become,

$$T_s(z) = T_1 - (\Delta T)\left(\frac{z}{d} + \frac{1}{2}\right) + \varepsilon\, \mathrm{Re}\left\{\frac{(\Delta T)\sinh\{\gamma(\frac{d}{2} - z)\}e^{i\omega t}}{\sinh(\gamma d)}\right\} \tag{5.2}$$

$$c_s(z) = c_1 + (\Delta T)\frac{k_T}{T_m}\left(\frac{z}{d} + \frac{1}{2}\right) + \varepsilon\, \mathrm{Re}\Big[\frac{k_T}{(\mathcal{D}-\lambda)}\left(\frac{\Delta T}{T_m}\right)\frac{1}{\sinh(\gamma d)}\sqrt{\mathcal{D}\lambda}$$

$$\times\left\{\left(\frac{\cosh\{\eta(\frac{d}{2} + z)\} - \cosh(\gamma d)\cosh\{\eta(\frac{d}{2} - z)\}}{\sinh(\eta d)}\right) - 1\right\}\Big], \tag{5.3a}$$

where

$$\gamma^2 = \frac{i\omega}{\lambda}, \tag{5.3b}$$

$$\eta^2 = \frac{i\omega}{\mathcal{D}}; \tag{5.3c}$$

and other symbols have their usual meaning as defined in Chapter II. At the onset of convection, fluid motion starts in the system. If $\vec{V}$ represents velocity at the onset and $\delta\rho$, δT, δc and δP, the fluctuations in mean density, temperature, concentration and pressure respectively because of fluid motion, the hydrodynamic equations for disturbances in linearized form are given by Eqs.(2.10)-(2.12) in Chapter II. The infinite extensions of bounding surfaces allow us to express these disturbances in terms of two-dimensional periodic waves in XY plane. The hydrodynamic

equations on non-dimensionalization are expressed as

$$(D^2-a^2)(D^2-a^2-\frac{\partial}{\partial\tau})w = Ra^2\,(\theta-kc), \qquad (5.4)$$

$$(D^2-a^2-\sigma\frac{\partial}{\partial\tau})\theta = -w\,\{1+f(z,\omega)e^{i\bar{\omega}\tau}\}, \qquad (5.5)$$

$$(D^2-a^2-\frac{\sigma}{s}\frac{\partial}{\partial\tau})c = (D^2-a^2)\theta-\frac{w}{s}[1-g(z,\omega)e^{i\bar{\omega}\tau}], \quad (5.6)$$

where

$$f(z,\omega) = \mathrm{Re}\left[\frac{\gamma d\cosh\{\gamma d\,(1/2-z)\}}{\sinh(\gamma d)}\right] \qquad (5.7a)$$

and

$$g(z,\omega) = -\frac{\sqrt{\sigma\lambda}}{(\sigma-\lambda)}\mathrm{Re}\left[\frac{\eta d\sinh\{\eta d\,(z+\frac{1}{2})\}\{1+\cosh(\gamma d)\}}{\sinh(\eta d)}.\right]. \qquad (5.7b)$$

By defining a non-dimensional mass current

$$j = c-\theta \qquad (5.8)$$

and dropping the correction in the differential equation for j, the linearized perturbation equations for the system of modulated binary liquids are

$$(D^2-a^2)(D^2-a^2-p)w = Ra^2(1-k)\theta - Ra^2k\,j, \qquad (5.9)$$

$$(D^2-a^2-\sigma p)\theta = -w[1+f(z,\omega)\,e^{i\bar{\omega}\tau}] \qquad (5.10)$$

$$(D^2-a^2-\frac{\sigma}{s}p)j = (D^2-a^2)\frac{\theta}{s}, \qquad (5.11)$$

where 'a' is non-dimensional wave number in horizontal plane and 'p' represents the rate of growth of fluctuations.

The realistic boundary conditions are

$$w = Dw = \theta = Dj = 0 \text{ at } z = \pm 1/2 . \qquad (5.12)$$

We proceed by assuming the amplitude of modulation to be small ($\varepsilon \ll 1$). In this case, we expand w, θ, j and R in powers of 'ε' as

$$w = w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots , \qquad (5.13)$$

$$\theta = \theta_o + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots , \qquad (5.14)$$

$$j = j_o + \varepsilon j_1 + \varepsilon^2 j_2 + \ldots , \qquad (5.15)$$

$$R = R_o + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots , \qquad (5.16)$$

Inserting these expansions in Eqs.(5.9)-(5.12) and collecting terms of the same order in 'ε', we find up to $O(\varepsilon^2)$

$$(D^2-a^2)(D^2-a^2-p)w_o = R_o a^2(1-k)\theta_o - R_o a^2 kj_o, \qquad (5.17a)$$

$$(D^2-a^2 - \sigma p)\theta_o = -w_o, \qquad (5.17b)$$

$$(D^2-a^2 - \frac{\sigma}{s} p)j_o = (D^2-a^2) \frac{\theta_o}{s} \qquad (5.17c)$$

with

$$w_o = Dw_o = \theta_o = Dj_o = 0 \text{ at } z = \pm \frac{1}{2} ; \qquad (5.17d)$$

$$(D^2-a^2)(D^2-a^2-p)w_1 = R_o a^2(1-k)\theta_1 - R_o a^2 k j_1$$
$$+R_1 a^2(1-k)\theta_o - R_1 a^2 kj_o, \qquad (5.18a)$$

$$(D^2-a^2- \sigma p)\theta_1 = -w_1 - f(z,\omega)e^{i\bar{\omega}\tau} w_o, \qquad (5.18b)$$

$$(D^2-a^2 - \frac{\sigma}{s}\,p)j_1 = (D^2-a^2)\,\frac{\theta_1}{s} \qquad (5.18c)$$

with

$$w_1 = Dw_1 = \theta_1 = Dj_1 = 0 \text{ at } z = \pm\frac{1}{2}\,; \qquad (5.18d)$$

and

$$(D^2-a^2)(D^2-a^2-p)w_2 = R_o a^2(1-k)\theta_2 - R_o a^2 k\, j_2$$

$$+ R_1 a^2(1-k)\theta_1 + R_2 a^2(1-k)\theta_o$$

$$- R_1 a^2\, kj_1 - R_2\, a^2\, kj_o, \qquad (5.19a)$$

$$(D^2-a^2-\sigma\, p)\theta_2 = -w_2 - f(z,\omega)e^{i\omega\tau}\, w_1, \qquad (5.19b)$$

$$(D^2-a^2 - \frac{\sigma}{s}\,p) = (D^2-a^2)\,\frac{\theta_2}{s} \qquad (5.19c)$$

with

$$w_2 = Dw_2 = \theta_2 = Dj_2 = 0 \text{ at } z = \pm\frac{1}{2}\,. \qquad (5.19d)$$

Equations (5.17a)-(5.17c) represent unmodulated system of the binary liquid mixtures.

5.3 Stationary Convection and Modulation

At the onset of stationary convection the fluctuations are time-independent (p=0) in the absence of external modulation. So, Eqs.(5.17a)-(5.17c) reduce to

$$(D^2-a^2)^2 w_o = R_o a^2(1-k)\theta_o - R_o a^2\, k\, j_o\,, \qquad (5.20a)$$

$$(D^2 - a^2)\theta_o = -w_o \,, \tag{5.20b}$$

$$(D^2 - a^2)j_o = (D^2 - a^2)\,\frac{\theta_o}{s} \tag{5.20c}$$

with boundary conditions given by Eq.(5.17d). To solve this system of differential equations, we follow Chandrasekhar's technique[7] and expand θ_o in a Fourier cosine series as

$$\theta_o = \sum_{n=0}^{\infty} A_n \cos(2n+1)\pi z \,. \tag{5.21}$$

As we have seen in Chapter II, one-mode truncation yields quite accurate results (within 7%); and, therefore, we determine the effect of modulation within one-mode approximation. We choose

$$\theta_o = A_o \cos\pi z \,. \tag{5.22}$$

With this ansatz on 'θ', the Eq.(5.20c) is solved with boundary conditions $Dj_o = 0$ at $z = \pm 1/2$. The solution is found to be

$$j_o = \frac{A_o}{s}\left[\frac{\pi}{a}\,\frac{\cosh(az)}{\sinh(a/2)} + \cos\pi z\right]. \tag{5.23}$$

Now operating by $(D^2 - a^2)$ on Eq.(5.20a) and using Eq.(5.20c), the former becomes

$$(D^2-a^2)^3 w_o = R_o a^2 \left(1 - k - \frac{k}{s}\right)(D^2-a^2)\theta_o \,. \tag{5.24}$$

As $\theta_o = 0$ at $z = \pm 1/2$, Eq.(5.20a) yields a boundary condition on w_o given by

$$(D^2 - a^2)^2 w_o = -R_o a^2 k J_o \text{ at } z = \pm \frac{1}{2} \qquad (5.25)$$

in addition to the boundary conditions $w_o = Dw_o = 0$ at $z = \pm 1/2$. Inserting Eq.(5.22) in Eq.(5.24), the latter becomes

$$(D^2-a^2)^3 w_o = - R_o a^2 (1 - k - \frac{k}{s})(\pi^2 + a^2) A_o \cos\pi z \ . \qquad (5.26)$$

The general solution of this equation is

$$w_o = A_o \Big[B_1 \cosh(az) + B_2 z \sinh(az) + B_3 z^2 \cosh(az) + R_o a^2 \frac{(1 - k - \frac{k}{s})}{(\pi^2 + a^2)^2} \cos\pi z\Big] \ . \qquad (5.27)$$

The boundary conditions (5.25) yield

$$B_3 = - \frac{kR_o}{s} \cdot \frac{\pi}{8a} \frac{1}{\sinh(a/2)} \ . \qquad (5.28a)$$

Other boundary conditions ($w_o = Dw_o = 0$ at $z = \pm 1/2$) determine the constants B_1 and B_2, which are

$$B_1 = - \frac{1}{2} (B_2 \tanh(a/2) + \frac{B_3}{2}) \ , \qquad (5.28b)$$

$$B_2 = \frac{2 R_o a^2 \pi}{(\sinh a + a)} \Big[\frac{(1 - k - \frac{k}{s})}{(\pi^2 + a^2)^2} + \frac{k}{8s} \cdot \frac{1}{a^3} \coth(a/2)\Big] \times \cosh(a/2) \ . \qquad (5.28c)$$

Inserting Eqs.(5.22) and (5.27) in Eq.(5.20b), we obtain

$$(\pi^2 + a^2) \cos\pi z = - \Big[B_1 \cosh(az) + B_2 z \sinh(az) + B_3 z^2 \cosh(az) + \frac{R_o a^2 (1 - k - \frac{k}{s})}{(\pi^2 + a^2)^2} \cos\pi z \Big] \ . \qquad (5.29)$$

Multiplying throughout with $\cos\pi z$, integrating from $z = -1/2$ to $z = +1/2$, using Eqs. (5.28a) - (5.28c) and simplying we obtain

$$R_o = \frac{(\pi^2 + a^2)^3 / a^2}{[(1 - k - \frac{k}{s}) G_1(a) + \frac{k}{s} G_2(a)]}, \tag{5.30}$$

where

$$G_1(a) = 1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2 + a^2)^2(\sinh a + a)} \tag{5.31a}$$

and $$G_2(a) = \frac{\pi^2}{a^2} [1 + \frac{(\pi^2 - 3a^2)}{(\pi^2 + a^2)} \times$$

$$\times \coth(a/2) - \frac{(1+\cosh a)}{(\sinh a + a)} \coth a/2].$$

(5.31b)

The expression for R_o [Eq. (5.30)], when minimized with respect to the 'a', yields minimum value of the wavenumber

a_c and, thereof, critical value of Rayleigh number (R_{oc}) in the absence of modulation. Corrections due to the modulation are obtained by requiring that Eqs.(5.18a)-(5.18d) and Eqs.(5.19c)-(5.19d) must have solutions compatible with that of Eqs.(5.17a)-(5.17d). The solvability criterion for Eqs.(5.18a)-(5.18c) yields

$$\frac{1}{T}\int_0^T (\tilde{w}_o \quad \tilde{\theta}_o \quad \tilde{j}_o) \begin{pmatrix} R_1 a^2(1-k)\theta_o - R_1 a^2 k j_o \\ -f(z,\omega)\, e^{i\omega\tau}\, w_o \\ 0 \end{pmatrix} d\tau = 0, \tag{5.32}$$

where $\tilde{w}_o$, $\tilde{\theta}_o$, $\tilde{j}_o$ are left vectors of differential operator of Eqs.(5.17a)-(5.17c) and 'T', the time period of the modulation. The left vectors in the absence of modulation are given by equations

$$(D^2 - a^2)^2 \tilde{w}_o + \tilde{\theta}_o = 0 \tag{5.33a}$$

$$-R_o a^2(1-k)\tilde{w}_o + (D^2-a^2)\tilde{\theta}_o - \frac{1}{s}(D^2-a^2)\tilde{j}_o = 0 \tag{5.33b}$$

and

$$R_o a^2 k \tilde{w}_o + (D^2 - a^2)\tilde{j}_o = 0 \tag{5.33c}$$

with boundary conditions

$$\tilde{w}_o = D\tilde{w}_o = \tilde{\theta}_o = D\tilde{j}_o = 0 \text{ at } z = \pm\frac{1}{2}\,. \tag{5.33d}$$

We solve Eqs.(5.33a)-(5.33c) by making an ansatz on θ_o (as done before), which is

$$\theta_0 = A_0 \cos\pi z . \tag{5.34}$$

Inserting this in the Eq.(5.33a) and solving the resulting equation with boundary conditions on $\tilde{w}_0$, we find

$$\tilde{w}_0 = A_0[P \cosh(az) + Qz \sinh(az) - \frac{\cos\pi z}{(\pi^2 + a^2)^2}], \tag{5.35}$$

where

$$P = -\frac{Q}{2} \tanh(a/2) \tag{5.36a}$$

and

$$Q = -\frac{2\pi \cosh(a/2)}{(\pi^2+a^2)^2(\sinh a+a)} . \tag{5.36b}$$

The evaluation of $\tilde{j}_0$ is not needed as the term due to modulation in the equation for $\tilde{j}_0$ is neglected. Now from Eq.(5.32), we obtain

$$R_1 a^2(1-k)<\tilde{w}_0|\theta_0> - R_1 a^2 k <\tilde{w}_0|j_0> - <\tilde{\theta}_0 | \overline{f(z,\omega)^{i\omega\tau}} w_0> = 0, \tag{5.37}$$

where, bars denote time average over one cycle of modulation. As $f(z,\omega)$, $\tilde{\theta}_0$, $\tilde{w}_0$ and w_0 are time independent, the time averaging of the last term of Eq.(5.37) vanishes. Therefore, it leads to

$$R_1 = 0 , \tag{5.38}$$

which means there is no correction to the Rayleigh number up to $O(\varepsilon)$. To obtain the explicit forms of w_1 and j_1 upto

$O(\varepsilon)$, $f(z,\omega)w_o(z)$ is expanded in Fourier series and only the first term is retained. Thus,

$$f(z,\omega)w_o(z) = A_o\, f(z,\omega)\, \bar{w}_o(z) \simeq A_o\, F(\omega)\, \cos\pi z\ ,$$

where $\bar{w}_o(z) = \dfrac{w_o(z)}{A_o}$, (5.39a)

and

$$F(\omega) = 2\int_{-1/2}^{+1/2} f(z,\omega)\bar{w}_o \cos\pi z\, dz$$

$$= 2\,\mathrm{Re}\int_{-1/2}^{+1/2} \frac{\gamma d \cosh\{\gamma d(\frac{1}{2} - z)\}}{\sinh(\gamma d)} [B_1\cosh(az) + B_2 z\sinh(az) + B_3 z^2 \cosh(az) + $$

$$+ \frac{R_o a^2 (1 - k - \frac{k}{s})A_o \cos\pi z}{(\pi^2 + a^2)^2}] \cos\pi z\, dz$$

$$= 2\,\mathrm{Re}\,\frac{\gamma d}{\sinh(\gamma d)}[B_1 \int_{-1/2}^{+1/2} \cosh(az)\cdot\cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos\pi z\, dz$$

$$+ B_2 \int_{-1/2}^{+1/2} z \sinh(az)\, \cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos\pi z\, dz$$

$$+ B_3 \int_{-1/2}^{+1/2} z^2 \cosh(az)\, \cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos\pi z\, dz$$

$$+ \frac{R_o\, a^2 (1 - k - \frac{k}{s} A_o)}{(\pi^2 + a^2)^2} \int_{-1/2}^{+1/2} \cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos^2\pi z\, dz\ .$$

(5.39b)

Performing integrations in the above equation, we obtain

$$F(\omega) = [\,B_1 \mathcal{J}_1 + B_2 \mathcal{J}_2 + B_3 \mathcal{J}_3 + \frac{R_o\, a^2(1-k-\frac{k}{s})A_o}{(\pi^2 + a^2)^2} \mathcal{J}_4\,], \tag{5.40}$$

where,

$$\mathcal{J}_1 = \frac{2\gamma d}{\sinh(\gamma d)} \int_{-1/2}^{+1/2} \cosh(az).\cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos\pi z \, dz$$

$$= \frac{\pi\gamma d}{\sinh(\frac{\gamma d}{2})} \left[\frac{\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}} + \frac{\cosh\{(a-\gamma d)/2\}}{\{(a-\gamma d)^2 + \pi^2\}} \right], \tag{5.41a}$$

$$\mathcal{J}_2 = \frac{2\gamma d}{\sinh(\gamma d)} \times \frac{d}{da} \left[\int_{-1/2}^{+1/2} \cosh(az)\, \cosh\{\gamma d(\tfrac{1}{2} - z)\}\cos\pi z\, dz \right]$$

$$= \frac{\pi\gamma d}{2\sinh(\frac{\gamma d}{2})} \quad \frac{\sinh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}} + \frac{\sinh\{(a-\gamma d)/2\}}{\{(a-\gamma d)^2 + \pi^2\}}$$

$$- \frac{4(a+\gamma d)\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}^2} - \frac{4(a-\gamma d)\cosh\{(a-\gamma d)/2\}}{\{(a-\gamma d)^2 + \pi^2\}^2}, \tag{5.42b}$$

$$\mathcal{J}_3 = \frac{2\gamma d}{\sinh(\gamma d)} \times \frac{d^2}{da^2} \left[\int_{-1/2}^{+1/2} \cosh(az)\cosh\{\gamma d(\tfrac{1}{2}-z)\}\cos\pi z\, dz \right]$$

$$= \frac{\pi\gamma d}{4\sinh(\frac{\gamma d}{2})} \left[\frac{\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}} - \frac{8(a+\gamma d)\sinh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}^2} \right.$$

$$\left. - \frac{8\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}^2} - \frac{8(a+\gamma d)^2\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2 + \pi^2\}^3} + \text{terms} \right.$$

with $\gamma d \to -\gamma d$ (5.42c)

and

$$\mathcal{I}_4 = \frac{2\gamma d}{\sinh(\gamma d)} \int_{-1/2}^{+1/2} \cosh\{\gamma d(\tfrac{1}{2} - z)\} \cos^2 \pi z \, dz$$

$$= \frac{4\pi^2}{(4\pi^2 + \gamma^2 d^2)} . \tag{5.42d}$$

The time dependence in w_1, θ_1 and j_1 can be expressed as

$$w_1(z,t) = W_1(z)\, e^{i\bar{\omega}\tau} , \tag{5.43a}$$

$$\theta_1(z,t) = \Theta_1(z)\, e^{i\bar{\omega}\tau} , \tag{5.43b}$$

$$j_1(z,t) = J_1(z)\, e^{i\bar{\omega}\tau} , \tag{5.43c}$$

Then Eqs.(5.18a)-(5.18d) with $R_1 = 0$ become

$$(D^2-a^2)(D^2-b^2)W_1(z) = R_o a^2(1-k)\Theta_1(z) - R_o a^2 k\, J_1(z), \tag{5.44a}$$

$$(D^2-b_1^2)\Theta_1(z) = -W_1(z) - f(z,\omega)\, w_o, \tag{5.44b}$$

$$(D^2-b_2^2)\, J_1(z) = (D^2-a^2) \frac{\Theta_1(z)}{s} , \tag{5.44c}$$

$$W_1(z) = DW_1(z) = \Theta_1(z) = D\, J_1(z) = 0 \text{ at } z = \pm \tfrac{1}{2} , \tag{5.44d}$$

where

$$b^2 = a^2 + i\bar{\omega}, \tag{5.45a}$$

$$b_1^2 = a^2 + i\,\bar{\omega}\sigma \tag{5.45b}$$

and

$$b_2^2 = a^2 + i\bar{\omega}\frac{\sigma}{s} . \tag{5.45c}$$

Consistent with the boundary conditions on $\Theta_1(z)$, we choose

$$\Theta_1(z) \simeq A_1 \cos\pi z \,. \tag{5.46}$$

Inserting this in Eq.(5.44c), we arrive at the equation

$$(D^2-b_2^2)\; J_1(z) = -\frac{A_1(\pi^2 + a^2)\cos\pi z}{s} \,. \tag{5.47}$$

The general solution of the above equation with the conditions $DJ_1(z) = 0$ at $z = \pm 1/2$ is

$$J_1(z) = \frac{(\pi^2+a^2)\,A_1}{s(\pi^2 + b_2^2)}\left[\frac{\cosh(b_2 z)}{b_2 \sinh(b_2/2)} + \cos\pi z\right]. \tag{5.48}$$

Using Eqs.(5.46) and (5.47) in the Eq.(5.44a), we have

$$(D^2-a^2)(D^2-b^2)W_1(z) = R_o a^2(1-k)A_1\cos\pi z - R_o a^2\,\frac{k}{s}\,\frac{(\pi^2+a^2)A_1}{(\pi^2+b_2^2)}\times$$
$$\times\left[\frac{\cosh(b_2 z)}{b_2 \sinh(b_2/2)} + \cos\pi z\right]. \tag{5.49}$$

The general solution of this equation is

$$W_1(z) = A_1\Bigg\{C_1 \sinh(az) + C_2 \cosh(az) + D_1\sinh(bz) + D_2\cosh(bz)$$
$$+ \frac{R_o a^2(1-k)\cos\pi z}{(\pi^2+a^2)(\pi^2+b^2)} - R_o a^2\,\frac{k}{s}\,\frac{(\pi^2+a^2)}{(\pi^2+b_2^2)}\Bigg[\frac{\cosh(b_2 z)}{\{b_2(b_2^2-a^2)(b_2^2-b_2^2)\cdot\sinh(b_2/2)\}} +$$
$$+ \frac{\cos\pi z}{(\pi^2+a^2)(\pi^2+b^2)}\Bigg]\Bigg\} \,. \tag{5.50}$$

Because of the symmetry of the system, we anticipate

$$C_1 = D_1 = 0 \,. \tag{5.51}$$

Now, the boundary conditions $W_1(z) = 0$ at $z = \pm 1/2$ gives

$$C_2 \cosh(a/2) + D_2 \cosh(b/2) = R_o a^2 \frac{k}{s} \frac{(\pi^2+a^2)}{(\pi^2+b_2^2)} \left[\frac{\coth(b_2/2)}{b_2(b_2^2-a^2)(b_2^2-b^2)} \right] \tag{5.52a}$$

and $DW_1(z) = 0$ at $z = \pm 1/2$ gives

$$a\, C_2 \sinh(a/2) + b\, D_2 \sinh(b/2) = \frac{R_o a^2(1-k)}{(\pi^2+a^2)(\pi^2+b^2)} +$$

$$+ \frac{R_o a^2 k \ (\pi^2+a^2)}{s\ (\pi^2+b_2^2)} \left\{ \frac{1}{(b_2^2-a^2)(b_2^2-b^2)} - \frac{1}{(\pi^2+a^2)(\pi^2+b^2)} \right\}. \tag{5.52b}$$

Eqs.(5.52a) and (5.52b) can be written in matrix form as

$$\begin{pmatrix} \cosh(a-2) & \cosh(b/2) \\ a\sinh(a/2) & b\sinh(b/2) \end{pmatrix} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \tag{5.53}$$

where

$$\alpha_1 = R_o a^2 \frac{k}{s} \frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)b_2} \frac{\coth(b_2/2)}{(b_2^2-a^2)(b_2^2 - b^2)} \tag{5.54a}$$

and

$$\beta_1 = \frac{R_o a^2(1-k)}{(\pi^2+a^2)(\pi^2+b^2)} + \frac{R_o a^2\, k\ (\pi^2+a^2)}{s\ (\pi^2 + b_2^2)} \times$$

$$\times \left[\frac{1}{(b_2^2-a^2)(b_2^2 - b^2)} - \frac{1}{(\pi^2+a^2)(\pi^2 + b^2)} \right]. \tag{5.54b}$$

From Eq. (5.53),

$$\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} = \begin{pmatrix} \cosh(a/2) & \cosh(b/2) \\ a\sinh(a/2) & b\sinh(b/2) \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \quad (5.55)$$

which yield

$$C_2 = \frac{\alpha_1 \, b \sinh(b/2) - \beta_1 \cosh(b/2)}{[b\sinh(b/2)\cosh(a/2) - a\sinh(a/2)\cosh(b/2)]} \quad (5.56a)$$

and $$D_2 = \frac{-\alpha_1 \, a \sinh(a/2) + \beta_1 \cosh(a/2)}{[b\sinh(b/2)\cosh(a/2) - a\sinh(a/2)\cosh(b/2)]} \quad (5.56b)$$

Redefining $C_2 = C$ (5.57a)

and $D_2 = D$, (5.57b)

Eq. (5.50) can be written as

$$W_1(z) = A_1[C\cosh(az) + D\cosh(bz) - \gamma_1 \cos\pi z + \gamma_2 \cosh(b_2 z)],$$

$$= A_1 \bar{W}_1(z) \quad (5.58)$$

where $\bar{W}_1(z) = [C\cosh(az) + D\cos(bz) - \gamma_1 \cos z + \gamma_2 \cosh(b_2)]$, (5.59a)

$$\gamma_1 = \frac{R_o a^2}{(\pi^2+a^2)(\pi^2+b^2)}\left[1 - k - \frac{k}{s}\,\frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)}\right] \quad (5.59b)$$

and

$$\gamma_2 = -\frac{R_o a^2 k(\pi^2 + a^2)}{s\, b_2\, (\pi^2 + b_2^2)} \times \frac{\pi}{(b_2^2-a^2)(b_2^2-b^2)\, \sinh(b_2/2)} \quad . \tag{5.59c}$$

Inserting Eqs.(5.39a), (5.46) and (5.58) in Eq.(5.44b)

$$(\pi^2 + b_1^2)A_1 \cos\pi z = W_1(z) + f(z,\omega)w_o \tag{5.60}$$

Multiplying the above equation throughout with $\cos\pi z$ and integrating from $z = -\frac{1}{2}$ to $z = +\frac{1}{2}$, we find

$$(\pi^2 + b_1^2)\frac{A_1}{2} = \int_{-1/2}^{+1/2} W_1(z)\cos\pi z\, dz + \int_{-1/2}^{+1/2} f(z,\omega)w_o\cos\pi z\, dz$$

$$= A_1 \int_{-1/2}^{1/2} \bar{W}_1(z)\cos\pi z\, dz + \frac{A_o}{2} F(\omega) \quad . \tag{5.61}$$

Redefining, $$\frac{F(\omega)}{R_o\, a^2} = \mathcal{F}(\omega), \tag{5.62a}$$

and $$\mathcal{K}(\omega) = \frac{K(\omega)}{R_o\, a^2} \tag{5.62b}$$

with $$K(\omega) = 2\int_{-1/2}^{1/2} \bar{W}_1(z)\cos\pi z\, dz \quad , \tag{5.62c}$$

Eq. (5.61) yields

$$\frac{A_1}{A_o} = \frac{R_o a^2 \mathcal{F}(\omega)/(\pi^2 + b_1^2)}{[1 - R_o a^2 \mathcal{K}(\omega)/(\pi^2 + b_1^2)]} \quad . \tag{5.63}$$

To obtain next order correction to the Rayleigh number, we

turn to Eqs.(5.19a)-(5.19c). The solvability condition at this order of 'ε' yields

$$\frac{1}{T}\int_0^T (\tilde{w}_o \quad \tilde{\theta}_o \quad \tilde{j}_o)\begin{pmatrix} R_2a^2(1-k)\theta_o - R_2a^2kj_o \\ -f(z,\omega)e^{i\bar{\omega}\tau} \quad w_1 \\ 0 \end{pmatrix} d\tau = 0 . \tag{5.64}$$

As $w_1 = W_1(z)e^{i\bar{\omega}\tau}$, the above equation after integrating over one cycle of modulation gives

$$R_2a^2(1-k)<\tilde{w}_o|\theta_o> - R_2a^2k<\tilde{w}_o|j_o> - \frac{1}{2}<\tilde{\theta}_o|f(z,\omega)W_1(z)> = 0$$

or,

$$R_2a^2 = \frac{1}{2}\,\mathrm{Re}\left[\frac{<\tilde{\theta}_o|f(z,\omega)\,W_1(z)>}{\{(1-k)<\tilde{w}_o|\theta_o> - k<\tilde{w}_o|j_o>\}}\right]. \tag{5.65}$$

Now using Eqs.(5.20b) and (5.22),

$$\begin{aligned}<\tilde{\theta}_o|f(z,\omega)\,W_1(z)> &= <\theta_o|f(z,\omega)\,W_1(z)> \\ &= \frac{1}{(\pi^2+a^2)}<W_o(z)|f(z,\omega)\,W_1(z)> \\ &= \frac{1}{(\pi^2+a^2)}<W_o(z)|f^*(z,\omega)\,W_1(z)> \\ &= \frac{R_oa^2\mathcal{F}^*(\omega)}{(\pi^2+a^2)}<\theta_o|W_1(z)>, \\ &= \frac{R_oa^2\mathcal{F}^*(\omega)A_1}{(\pi^2+a^2)}<\theta_o(z)|\bar{W}_1(z)>. \end{aligned} \tag{5.66}$$

Again from Eq.(5.35),

$$\tilde{w}_o = A_o \tilde{W}_o , \qquad (5.67a)$$

where

$$\tilde{W}_o = P \cosh(az) + Qz \sinh(az) - \frac{\cos \pi z}{(\pi^2+a^2)^2} . \qquad (5.67b)$$

Now, Eq.(5.65) becomes

$$R_2 a^2 = \frac{1}{2} \operatorname{Re}\left[\frac{<\tilde{\theta}_o(z) | f(z,\omega) \bar{W}_1(z)>}{<\tilde{W}_o(z) | \theta_o(z)>} \frac{A_1}{A_o} \times \right.$$

$$\left. \times \{1 - k(1 + \frac{<\tilde{W}_o(z) | j_o(z)>}{<\tilde{W}_o(z) | \theta_o(z)>})\}^{-1} \right] . \qquad (5.68)$$

Inserting Eqs.(5.66) and Eq.(5.63) in the above equation, we obtain

$$R_2 a^2 = \frac{1}{2} \operatorname{Re}\left(\frac{(R_o a^2)^2 |\mathcal{F}(\omega)|^2 <\theta_o(z)| \bar{W}_1(z)>}{(\pi^2+a^2)(\pi^2+b_1^2) <\tilde{W}_o(z) | \theta_o(z)>} \times \right.$$

$$\left. \times [1 - k\{1 + \frac{<\tilde{W}_o(z) | j_o(z)>}{<\tilde{W}_o(z) | \theta_o(z)>}\}^{-1}] \right) . \qquad (5.69)$$

In most of the liquids and over a significant parameter range in ^3He-^4He the ratio $\frac{k}{s}$ is very large. So, we will find correction in the limit $\frac{k}{s} >> 1$. In this case,

$$B_1 \to \frac{R_o a^2}{(\sinh a + a)} (\frac{k}{s}) [\frac{\sinh(a/2)}{(\pi^2 + a^2)^2} - \frac{(\sinh a - a)}{32 a^3 \sinh(a/2)}], \qquad (5.70a)$$

$$B_2 \to \frac{R_o\ a^2}{(\sinh a+a)} \cdot \left(\frac{k}{s}\right) \left[\frac{\cosh^2(a/2)}{4a^3 \sinh(a/2)} - \frac{2\cosh(a/2)}{(\pi^2 + a^2)^2} \right] \tag{5.70b}$$

and

$$\mathcal{F}(\omega)\Big|_{k/s \to \text{large}} \simeq \frac{k}{s} \left[\frac{\pi}{(\sinh a+a)} \left\{ \frac{\sinh(a/2)}{(\pi^2+a^2)^2} - \frac{(\sinh a-a)}{32a^3 \sinh(a/2)} \right\} \mathcal{I}_1 \right.$$

$$+ \frac{\pi}{(\sinh a+a)} \left\{ \frac{\cosh^2(a/2)}{4a^3 \sinh(a/2)} - \frac{2\cosh(a/2)}{(\pi^2 + a^2)^2} \right\} \mathcal{I}_2$$

$$\left. - \frac{\pi}{8a^3 \sinh(a/2)} \mathcal{I}_3 - \mathcal{I}_4 \right] . \tag{5.70c}$$

Writing

$$\mathcal{G} = \frac{1}{(\pi^2+a^2)^2} \left[\left(1 - k\frac{k}{s}\right) G_1(a) + \frac{k}{s} G_2(a) \right], \tag{5.71a}$$

$$\mathcal{G}\Big|_{k/s \to \text{large}} \simeq \frac{1}{(\pi^2 + a^2)^2} \cdot \frac{k}{s} \left[G_2(a) - G_1(a) \right] \tag{5.71b}$$

Now,

$$< \theta_o(z) \mid \bar{W}_1(z) > = \frac{1}{2} K = \frac{R_o a^2 \mathcal{K}}{2} , \tag{5.72}$$

where the explicit expression for $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{k}{s} \cdot \frac{1}{(\pi^2+b^2)(\pi^2+b_2^2)} < 4\pi^2 \left[\frac{(\pi^2+b^2)b\coth(b_2/2)\sinh(b/2)}{(b_2^2-a^2)(b_2^2-b^2)}\right.$$

$$(\pi^2+b^2) \times \left\{\frac{1}{(b_2^2-a^2)(b_2^2-b^2)} - \frac{1}{(\pi^2+a^2)(\pi^2+b^2)}\right\}\cosh(b/2)\Big]$$

$$\times \cosh(a/2) - 2\pi^2\left[\frac{(\pi^2+a^2)a\coth(b_2/2)\sinh(a/2)}{(b_2^2-a^2)(b_2^2-b^2)} - (\pi^2+a^2)\right.$$

$$\times\left\{\frac{1}{(b_2^2-a^2)(b_2^2-b^2)} - \frac{1}{(\pi^2+a^2)(\pi^2+b^2)}\right\} \times \cosh(a/2)\Big]\cosh(b/2)$$

$$- \frac{4\pi^2(\pi^2+a^2)(\pi^2+b^2)\coth(b_2/2)}{b_2(\pi^2+b_2^2)(b_2^2-a^2)(b_2^2-b^2)} - 1> \qquad (5.73)$$

So, from Eq.(5.30)

$$\frac{k}{s} R_o \Big|_{\frac{k}{s}\to \text{large}} = \frac{(\pi^2+a^2)^3/a^2}{[G_2(a) - G_1(a)]} . \qquad (5.74)$$

From Eq.(5.63),

$$\frac{A_1}{A_o}\Big|_{\frac{k}{s}\to\text{large}} \cong - \frac{\mathcal{F}(\omega)|_{k/s\to\text{large}}}{\mathcal{K}(\omega)} . \qquad (5.75)$$

We also have

$$<\tilde{W}_o(z)|\theta_o(z)> = - \frac{1}{2(\pi^2+a^2)^2} G_1(a) . \qquad (5.76)$$

Inserting Eqs.(5.70c), (5.72), (5.75) and (5.76) in Eq.(5.69),

we arrive at

$$\frac{k}{s} R_2 a^2 \simeq \frac{1}{4} \frac{(R_o a^2)^2 |(\omega)|^2}{(\pi^2+a^2) \langle \tilde{W}_o(z)| s j_o(z)\rangle}, \qquad (5.77)$$

where we have used the fact that $\langle \tilde{W}_o(z)| s j_o(z)\rangle$ is a term of order of $\frac{k}{s}$ and hence

$$[1 - k\{1 - \frac{2(\pi^2+a^2)\langle \bar{W}_o(z)| j_o(z)\rangle}{G_1(a)}\}]^{-1} \simeq \frac{G_1(a)}{2k(\pi^2+a^2)^2 \langle \bar{W}_o(z)| j_o(z)\rangle}$$

(5.78)

The sign of correction depends upon that of the term $\langle \bar{W}_o(z)| s j(z)\rangle$. In different range of parameter values this can be either positive or negative. Consequently, the modulation can either stabilize or destabilize the conduction state.

We have seen in Chapter II that the realistic boundaries makes convection with zero wavenumber possible. It is interesting to know whether modulation make the wave-number finite. The correction, if there is any, will be because of the second order correction to the Rayleigh number. It is given by

$$a_2 = -[(\frac{\partial R_2}{\partial a_o})/(\frac{\partial^2 R_o}{\partial a_o^2})]\Bigg|_{\substack{\frac{k}{s} \to \text{large} \\ a_o \to 0}} . \qquad (5.79)$$

In the limit $\frac{k}{s} >> 1$ and $a_o \to 0$, this correction is proportional

to a_o. For $a_o = 0$, this correction vanishes. This means the modulation doesn't affect the convection roll, although it changes the onset of convection.

5.4 Oscillatory Convection and Modulation

At the onset of oscillatory convection velocity field and fluctuations in temperature field and mass current are time dependent ($p \neq 0$). They can be expressed as

$$w_o^{os} = \mathrm{Re}\, W_o^{os}(z)\, e^{i \bar{\omega}_o \tau}, \tag{5.80a}$$

$$\theta_o^{os} = \mathrm{Re}\, \Theta_o^{os}(z)\, e^{i \bar{\omega}_o \tau}, \tag{5.80b}$$

$$j_o^{os} = \mathrm{Re}\, J_o^{os}(z)\, e^{i \bar{\omega}_o \tau}, \tag{5.80c}$$

where $\bar{\omega}_o$ is the angular frequency† of oscillations. Now Eqs.(5.17a)-(5.17c) become

$$(D^2-a^2)(D^2-b^2)W_o^{(os)} = R_o^{(os)} a^2(1-k)\Theta_o^{(os)} - R_o^{(os)}\, k\, J_o^{(os)}, \tag{5.81a}$$

$$(D^2-b_1^2)\,\Theta_o^{(os)} = -W_o^{(os)}, \tag{5.81b}$$

$$(D^2-b_2^2)\; J_o^{(os)} = (D^2-a^2)\,\frac{\Theta_o^{(os)}}{s}, \tag{5.81c}$$

where b, b_1, b_2 are as defined in Eqs.(2.54a)-(2.54c) (Chapter II). Eqs.(5.81a)-(5.81c) are same as Eqs.(2.51)-(2.53) and, hence, critical Rayleigh number the zeroth order

† Angular frequency ($\bar{\omega}_o$) is in non-dimensional form.

is same as that for unmodulated flow. That is,

$$R_o^{(os)} = R_{os} = \mathrm{Re}\left[\frac{(\pi^2 + b_1^2)/a^2}{\{(1-k-\frac{k}{s})\,\tilde{G}_1(a,\bar{\omega}) + \frac{k}{s}\,\tilde{G}_2(a,\bar{\omega})\}}\right] \tag{5.82}$$

with $\tilde{G}_1(a,\bar{\omega})$ and $\tilde{G}_2(a,\bar{\omega})$ given in Eqs.(2.66a)-(2.66b) [Chapter II].

The terms of $O(\varepsilon)$ are given in Eqs.(5.18a)-(5.18c) and can be rewritten as

$$(D^2-a^2)(D^2-a^2-p)\,w_1^{(os)} - R_o^{(os)}a^2(1-k)\,\theta_1^{(os)} + R_o^{(os)}a^2k\,j_o^{(os)} = R_1^{(os)}\,a^2(1-k)\,\theta_o^{(os)} - R_1^{(os)}a^2\,k\,j_o^{(os)}, \tag{5.83a}$$

$$w_1^{(os)} + (D^2-a^2-\sigma p)\,\theta_1^{(os)} = -\,w_o^{(os)}\,f(z,\omega)\,e^{i\bar{\omega}t}, \tag{5.83b}$$

$$-(D^2-a^2)\,\frac{\theta_1^{(os)}}{s} + (D^2-a^2-\frac{\sigma}{s}\,p)\,j_1^{(os)} = 0. \tag{5.83c}$$

The solvability criterion of Eqs.(5.83a)-(5.83c) demand that

$$\frac{1}{T}\int_0^T (\tilde{w}_o^{(os)}\;\tilde{\theta}_o^{(os)}\;\tilde{j}_o^{(os)})\begin{pmatrix} R_1^{(os)}a^2(1-k)\theta_o^{(os)} - R_1^{(os)}\,a^2\,k\,j_o^{(os)} \\ -w_o^{(os)}\,f(z,\omega)\,e^{i\bar{\omega}\tau} \\ 0 \end{pmatrix} d\tau = 0, \tag{5.84}$$

where 'T' is the time period of modulation and $\tilde{w}_o^{(os)}$, $\tilde{\theta}_o^{(os)}$ and $\tilde{j}_o^{(os)}$ are left vectors of the differential operator

$$L = \begin{pmatrix} (D^2-a^2)(D^2-a^2-p) & -R_o^{(os)}\, a^2(1-k) & R_o^{(os)}\, a^2\, k \\ 1 & (D^2-a^2-\sigma p) & 0 \\ 0 & -\frac{1}{s}(D^2-a^2) & (D^2-a^2-\frac{\sigma p}{s}) \end{pmatrix}. \tag{5.85}$$

As the convection is oscillatory at the onset, $\tilde{w}_o^{(os)}$, $\tilde{\theta}_o^{(os)}$ and $\tilde{j}_o^{(os)}$ can also be expressed

$$\tilde{w}_o^{(os)} = \text{Re}\ \tilde{W}_o^{(os)}(z)\ e^{i\bar{\omega}_o\tau}, \tag{5.86a}$$

$$\tilde{\theta}_o^{(os)} = \text{Re}\ \tilde{\Theta}_o^{(os)}(z)\ e^{i\bar{\omega}_o\tau}, \tag{5.86b}$$

$$\tilde{j}_o^{(os)} = \text{Re}\ \tilde{J}_o^{(os)}(z)\ e^{i\bar{\omega}_o\tau}, \tag{5.86c}$$

where $\tilde{W}_o^{(os)}$, $\tilde{\Theta}_o^{(os)}$ and $\tilde{J}_o^{(os)}$ are solutions of the equations

$$(D^2-a^2)(D^2-b^2)\ \tilde{W}_o^{(os)} + \tilde{\Theta}_o^{(os)} = 0, \tag{5.87a}$$

$$-R_o^{(os)}\ a^2(1-k)\ \tilde{W}_o^{os} + (D^2-b_1^2)\ \tilde{\Theta}_o^{(os)} - (D^2-a^2)\ \tilde{J}_o^{(os)} = 0 \tag{5.87b}$$

and

$$R_o^{(os)}\ a^2\ k\tilde{W}_o^{(os)} + (D^2-b_2^2)\ \tilde{J}_o^{(os)} = 0 \tag{5.87c}$$

with boundary conditions

$$D\tilde{J}_o^{(os)} = \tilde{W}_o^{(os)} = D\ \tilde{W}_o^{(os)} = \tilde{\Theta}_o^{(os)} = 0 \text{ at } z = \pm\frac{1}{2} \tag{5.87d}$$

It is evident from Eq.(5.84) that we do not need explicit

expression for $\tilde{J}_0^{(os)}$ to determine $R_1^{(os)}$. We determine expression for $\tilde{W}_0^{(os)}$ by making an ansatz on $\tilde{\Theta}_0^{(os)}$, which is

$$\tilde{\Theta}_0^{(os)} = \tilde{A}_{os} \cos \pi z \quad . \tag{5.88}$$

Now Eq.(5.87a) with boundary conditions on $\tilde{W}_0^{os}$ [Eq.(5.87d)] yield

$$\tilde{W}_0^{(os)} = \tilde{P} \cosh(az) + \tilde{Q} \cosh(bz) - \frac{\tilde{A}^{(os)} \cos \pi z}{(\pi^2+a^2)(\pi^2+b^2)}, \tag{5.89}$$

where

$$P = \frac{\tilde{A}_{os}}{[b \sinh(b/2) \cosh(a/2) - a \sinh(a/2) \cosh(b/2)]} \times$$

$$\times \frac{\cosh(b/2)}{(\pi^2+a^2)(\pi^2 + b^2)} \tag{5.90a}$$

and

$$\tilde{Q} = -\tilde{P} \frac{\cosh(a/2)}{\cosh(b/2)} \quad . \tag{5.90b}$$

Now $O(\varepsilon)$ correction to the Rayleigh number due to modulation is evaluated by setting the system for parametric resonance. For this the frequency of external modulation is set equal to twice of that of the system. That is,

$$\bar{\omega} = 2\bar{\omega}_0 \quad . \tag{5.91}$$

Therefore, from Eq.(5.84), using Eqs.(5.81a)-(5.81c) and Eqs.(5.86a)-(5.86b), we find

$$\frac{1}{T} \cdot \int_0^T R_1^{(os)} a^2(1-k) < \tilde{W}_0^{(os)} | \Theta_0^{(os)} > \cos^2 \bar{\omega}_0 \tau . d\tau$$

$$- \frac{1}{T} \int_0^T R_1^{(os)} a^2 k < \tilde{W}_0^{(os)} | J_0^{(os)} > \cos^2 \bar{\omega}_0 \tau . d\tau$$

$$- \frac{1}{T} \int_0^T < \tilde{\Theta}_0^{(os)} | W_0^{(os)} f(z,\omega) > \cos^2 \bar{\omega}_0 \tau . \cos 2\bar{\omega}_0 \tau . d\tau = 0$$

or

$$R_1^{(os)} a^2(1-k) < \tilde{W}_0^{(os)} | \Theta_0^{(os)} > - R_1^{(os)} a^2 \frac{k}{s} < \tilde{W}_0^{(os)} | s J_0^{(os)} >$$

$$= \frac{1}{2} < \tilde{\Theta}_0^{(os)} | W_0^{(os)} f(z,\omega) > . \qquad (5.92)$$

Here $J_0^{(os)}$ and $W_0^{(os)}$ are same as J_{os} and W_{os} respectively in Eqs.(2.55) and (2.59) (Chapter II). To evaluate the integral at r.h.s. of Eq.(5.92), we expand $W_0^{(os)} f(z,\omega)$ in Fourier cosime series and retain only one term

$$W_0^{(os)} f(z,\omega) = F^{(os)}(\omega) \cos \pi z , \qquad (5.93)$$

where

$$F^{(os)}(\omega) = 2 \int_{-1/2}^{+1/2} W_0^{(os)} f(z,\omega) \cos \pi z \, dz \qquad (5.94a)$$

$$= 2 \int_{-1/2}^{+1/2} [C_1 \cosh(az) + C_2 \cosh(bz) + C_3 \cosh(b_2 z) +$$

$$+ \frac{R_0^{(os)} a^2 \bar{\alpha} A_{os} \cos\pi z}{(\pi^2+a^2)(\pi^2+b^2)(\pi^2+b_2^2)}] \gamma d \frac{\cosh\{\gamma d(\frac{1}{2} - z)\}}{\sinh(\gamma d)} . \cos\pi z \, dz .$$

$$(5.94b)$$

That is,

$$F^{(os)}(\omega) = \frac{\pi\gamma d\ C_1}{\sinh(\gamma d/2)}\left[\frac{\cosh\{(a+\gamma d)/2\}}{\{(a+\gamma d)^2+\pi^2\}} + \frac{\cosh\{(a-\gamma d)/2\}}{\{(a-\gamma d)^2+\pi^2\}}\right] +$$

$$+ \frac{\pi\gamma\ d\ C_2}{\sinh(\gamma d/2)}\left[\frac{\cosh\{(b+\gamma d)/2\}}{\{(b+\gamma d)^2+\pi^2\}} + \frac{\cosh\{(b-\gamma d)/2\}}{\{(b-d)^2+\pi^2\}}\right] +$$

$$+ \frac{\pi\gamma\ d\ C_3}{\sinh(\ d/2)}\left[\frac{\cosh\{(b_2+\gamma d)/2\}}{\{(b_2+\gamma d)^2+\pi^2\}} + \frac{\cosh\{(b_2-\gamma d)/2\}}{\{(b_2-\gamma d)^2+\pi^2\}}\right] +$$

$$+ \frac{R_o^{(os)}\ a^2\ \bar{\alpha}}{(\pi^2+a^2)(\pi^2+b^2)(\pi^2+b_2^2)} \cdot \frac{4\pi^2}{(4\pi^2+\gamma^2\ d^2)}. \tag{5.94c}$$

We denote

$$F^{(os)}(\omega)\Big|_{\omega=2\omega_o} = F_{os} .$$

Now we evaluate other integrals of Eq.(5.92)

$$<\tilde{W}_o^{(os)}|\Theta_o^{(os)}> = \int_{-1/2}^{+1/2} [\tilde{P}\cosh(az) + \tilde{Q}\cosh(bz) -$$

$$- \frac{\tilde{A}_{os}\cos\pi z}{(\pi^2+a^2)(\pi^2+b^2)}]\, A_{os}\cos\pi z\ dz$$

$$= A_{os}\tilde{A}_{os}\left[\frac{2\tilde{P}\cosh(a/2)}{(\pi^2+a^2)} + \frac{2\pi\tilde{Q}\cosh(b/2)}{(\pi^2+b^2)}\right.$$

$$\left. - \frac{1}{2(\pi^2+a^2)(\pi^2+b^2)}\right].$$

Inserting Eqs.(5.90a)-(5.90b) in the previous equation, we arrive at

$$< \tilde{W}_o^{(os)} | \Theta_o^{(os)} > = - \frac{\tilde{G}_1}{2} . \qquad (5.95a)$$

Further, from Eqs.(5.89)and(2,55) [Chapter II] by making use of Eqs.(5.90a) and (5.90b) we obtain

$$< \tilde{W}_o^{(os)} | s\, J_o^{(os)} > = - [H + \frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)} \frac{\tilde{G}_1}{2}], \qquad (5.95b)$$

where G_1 is given in Eq.(2.66a) and

$$H = \frac{\pi^2}{b_2(\pi^2+b^2)(\pi^2+b_2^2)} < \frac{1}{\{b \sinh(b/2)\cosh(a/2)-a\sinh(a/2)\cosh(b/2)\}} \times$$

$$\times [\{\frac{\sinh\frac{(b_2+b)}{2}}{(b_2+b)} + \frac{\sinh\frac{(b_2-b)}{2}}{(b_2-b)}\} \frac{\cosh(a/2)}{\sinh(b_2/2)} -$$

$$- \{\frac{\sinh\frac{(b_2+a)}{2}}{(b_2+a)} + \frac{\sinh\frac{(b_2-a)}{2}}{(b_2-a)}\} \frac{\cosh(b/2)}{\sinh(b_2/2)}]+$$

$$+ 2 \frac{\coth(b_2/2)}{(\pi^2 + b_2^2)} >. \qquad (5.96)$$

Inserting Eqs.(5.93), (5.95a) and (5.95b) in Eq.(5.92) we find explicit expression for the first order correction in Rayleigh number due to modulation as

$$R_1^{(os)} = - \frac{1}{2} \text{Re} < \frac{F_{os}}{[1 - k - \frac{k}{s} \frac{(\pi^2 + a^2)}{(\pi^2 + b_2^2)}] \tilde{G}_1 - 2 \frac{k}{s} H} >. \qquad (5.97)$$

This expression is derived for $\omega = 2\omega_o$.

It is worthwhile to look for the asymptotic limit ($\omega \to 0$). In this limit, from Eq.(5.94b) and Eqs.(2.60) - (2.61c)

$$F_{os}(\omega)\Big|_{\omega \to 0} \simeq \frac{R_o^{(os)} \; a^2}{(\pi^2 + a^2)^2} \; (1-k-k/s) \; . \qquad (5.98a)$$

Further,

$$H(\omega)\Big|_{\omega \to 0} \simeq \frac{2^2}{(\pi^2 + a^2)^3} \; \frac{\coth(a/2)}{a} \qquad (5.98b)$$

and

$$\tilde{G}_1(\omega)\Big|_{\omega \to 0} \simeq G(0) = 1 - \frac{16\pi^2 a \cosh^2(a/2)}{(\pi^2+a^2)^2(a+\sinh a)} \; . \qquad (5.98c)$$

Inserting Eqs.(5.98a) - (5.98c) in Eq.(5.97), we obtain

$$R_1^{(os)}\Big|_{\omega \to 0} = - \frac{1}{2} \left[\frac{(1 - k - \frac{k}{s}) \, R_o^{(os)}}{(1-k-\frac{k}{s})G(0) - \frac{k}{s} \, \frac{4\pi^2}{(\pi^2+a^2)} \, \frac{\coth(a/2)}{a}} \right] . \qquad (5.99)$$

In case of free boundaries G(0) becomes equal to unity and other term with 'k/s' does not exist. Thus $R_{1f}^{(os)}\Big|_{\omega \to 0} = - \frac{1}{2} R_o^{(os)}$, which shows a destabilization. The result is the same as that obtained by Agarwal et.al.[8] for binary liquids with free boundaries using a Lorenz-like truncation of the system. However, in case of rigid boundaries, the explicit expression is not reported earlier. The correction in the limit $k/s \gg 1$, which is true for most

of the liquids and over a significant range in $^3He-^4He$, is found to be

$$R_1^{(os)}\Big|_{\substack{\omega \to 0 \\ \frac{k}{s} \to \text{large}}} = -\frac{1}{2}\left[\frac{R_o^{(os)}}{G(0) + \frac{4\pi^2}{(\pi^2+a^2)}\,\frac{\coth(a/2)}{a}}\right],$$

which shows a destabilization.

REFERENCES

1. See references (1-13) in Chapter II.

2. Gershuni, G.Z. and Zhukovitskii, "Convective Stability of Incompressible Fluids" [(Israel Programme for Scientific Translations, Jerusalem (1976)] .

3. Behringer, R.P., Rev. Mod. Phys. 57, 657 (1985).

4. Veronis, G., J. Mar. Res. 23, 1 (1965).

5. Schechter, R.S., Prigogine and Hamm, J.R., Phys. Fluids 15, 379 (1972).

6. Brand, H., Hohenberg, P.C. and Steinberg, V., Phys. Rev. A27, 591 (1983) and also Phys. Rev. A30, 2548 (1984).

7. Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability", [Clarendon Press, Oxford Press,] 1961 .

8. Agarwal, A., Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. B30,6458 (1984).

CHAPTER VI*

INSTABILITY IN MODULATED COUETTE-TAYLOR FLOW

6.1 Introduction

The stability of modulated Couette-Taylor flow is a problem of long standing. The reported theoretical[1-5] and experimental results[6-7] do not agree over the entire range of frequency of modulation. The full hydrodynamic equations for the flow have been studied analytically by Hall[1] and numerically by Riley and Laurence[3]. While Hall[1] finds destabilization of flow due to modulation for all frequencies, Riley and Laurence[3] notice, for high amplitudes of modulation, destabilization in low frequency regime and stabilization in high frequency regime. For low amplitudes they do not notice the stabilization at high frequencies. The experiments of Donnelly[6] show stabilization at all frequencies, while the more accurate experiment of Thompson[7] shows a destabilization at low frequencies and a stabilization at the higher values. The success of Chandrasekhar's technique[8] in the analysis of convective instabilities in Rayleigh-Bénard flows (both modulated and unmodulated) and in unmodulated Couette-Taylor flow motivates us to investigate the stability of modulated Couette-Taylor flow with proper rigid boundary conditions using this technique.

* Contents of this chapter have been accepted in Phys. Rev. A.

6.2 Hydrodynamics of the Couette-Taylor Flow

A fluid layer is confined between two coaxial vertical cylinders of infinite length, the outer ~~one~~ of which is fixed and the inner one is rotating with uniform angular velocity Ω_o about the common axis. Navier-Stokes equations for viscous incompressible fluid, in cylindrical polar coordinates, following Chandrasekhar's nomenclature read

$$\frac{\partial U_r}{\partial t} + (\vec{U}.\vec{\nabla})U_r - \frac{U_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\nabla^2 U_r - \frac{2}{r^2}\frac{\partial U_\varphi}{\partial \varphi} - \frac{U_r}{r^2}\right), \tag{6.1a}$$

$$\frac{\partial U_\varphi}{\partial t} + (\vec{U}.\vec{\nabla})U_\varphi + \frac{U_r U_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \nu\left(\nabla^2 U_\varphi + \frac{2}{r^2}\frac{\partial U_r}{\partial \varphi} - \frac{U_\varphi}{r^2}\right), \tag{6.1b}$$

$$\frac{\partial U_z}{\partial t} + (\vec{U}.\vec{\nabla})\, U_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 U_z, \tag{6.1c}$$

with

$$(\vec{U}.\vec{\nabla}) \equiv U_r \frac{\partial}{\partial r} + \frac{U_\varphi}{r}\frac{\partial}{\partial \varphi} + U_z \frac{\partial}{\partial z} \tag{6.1d}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{6.1e}$$

Here U_r, U_φ and U_z are respectively radial, azimuthal and vertical components of the velocity field $(\vec{U})$; ν is the kinematic viscosity. In addition to these equations, we have the equation of continuity,

$$\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\varphi}{\partial \varphi} + \frac{\partial U_z}{\partial z} = 0 . \qquad (6.2)$$

The steady state for the system of Eqs.(6.1a)-(6.1c) is described by

$$U_r = U_z = 0 \text{ and } U_\varphi = V(r) \qquad (6.3)$$

if

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{V^2}{r} \qquad (6.4a)$$

and

$$\nu\left(\nabla^2 V - \frac{V}{r^2}\right) = \nu\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right)V = 0 . \qquad (6.4b)$$

The expression for V can be found out by solving Eq. (6.4b) with proper boundary conditions on it.

Now we consider disturbance in the steady state and assume various perturbations axismmetric and independent of φ. If u_r, u_φ and u_z be the variation in steady state values of r, φ and z component of the velocity respectively and δP, the fluctuation in pressure field, the hydrodynamic equations in linearized form are found to be

$$\frac{\partial u_r}{\partial t} - 2\frac{V}{r}u_\varphi = -\frac{1}{\rho}\frac{\partial(\delta P)}{\partial r} + \nu\left(\nabla_1^2 u_r - \frac{u_r}{r^2}\right), \qquad (6.5a)$$

$$\frac{\partial u_\varphi}{\partial t} + \left(\frac{dV}{dr} + \frac{V}{r}\right)u_r = \nu\left(\nabla_1^2 u_\varphi - \frac{u_\varphi}{r^2}\right), \qquad (6.5b)$$

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial(\delta P)}{\partial z} + \nu\nabla_1^2 u_z , \qquad (6.5c)$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} . \tag{6.5d}$$

The equation for continuity takes the form

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 . \tag{6.6}$$

The disturbance are analyzed into normal modes and the solutions of Eqs.(6.5a)-(6.5c) are written in the form

$$u_r = \bar{u}(r,t)\, e^{i\bar{k}z} , \tag{6.7a}$$

$$u_\varphi = \bar{v}(r,t)\, e^{i\bar{k}z} , \tag{6.7b}$$

$$u_z = \bar{w}(r,t)\, e^{i\bar{k}z} , \tag{6.7c}$$

$$\delta P = \bar{p}(r,t)\, e^{i\bar{k}z} , \tag{6.7d}$$

where, $\bar{k}$ is the wave number of the disturbance in the axial direction. Inserting Eqs.(6.7a)-(6.7d) in Eqs.(6.5a)-(6.6) and then eliminating $\bar{w}$ and $\bar{p}$ from resulting equations we arrive at following equations:

$$\frac{\nu}{\bar{k}^2}\left[\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right) - \bar{k}^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right]\left[\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right) - \bar{k}^2\right]\bar{u} = 2\,\frac{V}{r}\,\bar{v} \tag{6.8a}$$

and

$$\nu\left[\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right) - \bar{k}^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right]\bar{v} = \left(\frac{dV}{dr} + \frac{V}{r}\right)\bar{u} . \tag{6.8b}$$

We proceed further by considering the narrow gap approximation - the separation (d) between the cylinders is small compared to the radius r_1 of inner cylinder. In this approximation, $(\frac{d}{dr} + \frac{1}{r})$ can be replaced by $\frac{d}{dr}$. Also Ω ($= \frac{V}{r}$), in the absence of external modulation can be written as

$$\Omega = \Omega_o \left[1 - \frac{(r - r_1)}{d}\right]. \tag{6.9}$$

Thus, the relevant hydrodynamic equations for axisymmetric flow in non-dimensional form using Eqs.(6.8a)-(6.8b) after proper rescaling are

$$\left(\frac{\partial^2}{\partial \xi^2} - a^2 - \frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2}{\partial \xi^2} - a^2\right) u = (1 - \xi)\, v \tag{6.10a}$$

and

$$\left(\frac{\partial^2}{\partial \xi^2} - a^2 - \frac{\partial}{\partial \tau}\right) v = - T a^2 u , \tag{6.10b}$$

where u and v are the dimensionless radial and azimuthal components of the velocity fluctuation, a is the non-dimensional wave number of the roll. In narrow gap approximation, there is only one variable $\xi = (r-r_1)/d$, which ranges from 0 to 1. T is the dimensionless Taylor number defined as

$$T = \frac{2\, \Omega_o^2\, d^3\, \bar{r}}{\nu^2} , \tag{6.11}$$

where $\bar{r} = (r_1 + r_2)/2$

is the mean radius of the system, ν , the kinematic viscosity

and τ, dimensionless time parameter.

Now, we start modulating the angular velocity of the inner cylinder as

$$\Omega = \Omega_o (1 + \varepsilon \cos\omega\tau), \qquad (6.12)$$

where ε and ω are amplitude and frequency of the modulation respectively. In the steady state, the flow has only azimuthal component of the velocity given as

$$\bar{V} = 1- \xi + \varepsilon \mathrm{Re} \frac{\sinh\{\alpha d(1-\xi)\}}{\sinh(\alpha d)} e^{i\omega\tau}, \qquad (6.13)$$

where

$$\alpha^2 = \frac{i\omega}{\nu} .$$

The linearized equations for the modulated flow in non-dimensional form are

$$\left(\frac{\partial^2}{\partial\xi^2} - a^2 - \frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2}{\partial\xi^2} - a^2\right) u = \left[1 - \xi + \varepsilon \mathrm{Re} \frac{\sinh\{\alpha d(1-\xi)\}}{\sinh(\alpha d)} e^{i\omega\tau}\right.$$

$$(6.14a)$$

$$\left(\frac{\partial^2}{\partial\xi^2} - a^2 - \frac{\partial}{\partial\tau}\right) v = - Ta^2\left[1 + \varepsilon \mathrm{Re} \frac{\alpha d}{\sinh(\alpha d)} \cosh\{\alpha d(1-\xi)\} e^{i\omega\tau}\right] u$$

$$(6.14b)$$

The difference from the hydrodynamic equations for the modulated Rayleigh-Bénard problem lies in the extra terms on the right hand side (rhs) of Eq.(6.14a) as compared to rhs of

Eq. (4.12). The qualitative difference in the structure of Eq. (6.14a) from that of Eq. (4.12) persists even in the absence of modulation. While the modulation affects only one of the equations in Rayleigh-Bénard system, the effect shows up in both equations in Couette-Taylor configuration and that causes a qualitative difference in the final result.

6.3 Mathematical Analysis

Equations (6.14a) and (6.14b) are solved perturbatively, as in Chapter IV by employing the modulation amplitude ε as the small parameter. Expanding variables u and v and the control parameter T as

$$u = u_o + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\ldots, \tag{6.15a}$$

$$v = v_o + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots\ldots, \tag{6.15b}$$

$$T = T_o + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots\ldots, \tag{6.15c}$$

and then inserting in Eqs.(6.14a) and (6.14b) and equating the identical powers of ε, we have up to $O(\varepsilon^2)$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})(D^2 - a^2)\, u_o = (1-\xi)v_o\ , \tag{6.16a}$$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})\, v_o = - T_o a^2 u_o\ ; \tag{6.16b}$$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})(D^2-a^2)u_1 = (1-\xi)v_1 + \mathrm{Re}\ g(\xi)v_o\, e^{i\omega\tau}, \tag{6.17a}$$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})v_1 = - T_o a^2 u_1 - T_1 a^2 u_o - T_o a^2 u_o \mathrm{Re}\, f(\xi) e^{i\omega\tau}; \tag{6.17b}$$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})(D^2 - a^2)u_2 = (1-\xi)v_2 + \mathrm{Re}\, g(\xi)\, v_1\, e^{i\omega\tau}, \tag{6.18a}$$

$$(D^2 - a^2 - \frac{\partial}{\partial\tau})v_2 = -T_o a^2 u_2 - T_2 a^2 u_o - T_1 a^2 u_1 - T_o a^2 u_1 \mathrm{Re}\, f(\xi)\, e^{i\omega\tau}; \tag{6.18b}$$

where

$$g(\xi) = \frac{\sinh\{\alpha d(1-\xi)\}}{\sinh(\alpha d)}, \tag{6.19a}$$

$$f(\xi) = \frac{\alpha d}{\sinh(\alpha d)} \cosh\{\alpha d(1-\xi)\} \tag{6.19b}$$

and

$$D \equiv \frac{d}{d\xi}. \tag{6.19c}$$

The boundary conditions require that all three components of the velocity must vanish at the rigid walls. This implies that

$$u = v = \frac{du}{d\xi} = 0 \text{ at } \xi = 0 \text{ and } 1. \tag{6.20}$$

We begin with the O(1) equations, Eqs.(6.16a) and (6.16b) which represent unmodulated flow. The boundary conditions at this order one

$$v_o = u_o = \frac{du_o}{d\xi} = 0 \text{ at } \xi = 0 \text{ and } 1. \tag{6.21}$$

Here, the only requirement on the azimuthal component (v_o)

of the velocity is vanishing at the walls. So, we proceed, as before, by expanding v_o in Fourier series and truncating the expansion at first term. This leads to making an ansatz

$$v_o \simeq \sin \pi\xi, \tag{6.22}$$

where the undetermined amplitude is chosen to be unity. Using Eq.(6.22) in Eq.(6.16a) and noting that there is no time dependence in unmodulated flow, we find

$$(D^2 - a^2)^2 u_o = (1-\xi) \sin \pi\xi. \tag{6.23}$$

The general solution of the above equation is

$$u_o(\xi) = A_o^* \sinh(a\xi) + B_o^* \cosh(a\xi) + C_o^*\xi\sinh(a\xi)$$
$$+ D_o^* \xi\cosh(a\xi) + \frac{(1-\xi)\sin(\pi\xi)}{(\pi^2 + a^2)^2} - \frac{4\pi}{(\pi^2+a^2)^3} \cos(\pi\xi), \tag{6.24}$$

where A_o^*, B_o^*, C_o^* and D_o^* are determined by the boundary conditions on u_o. They are found to be

$$A_o^* = - \frac{4\pi(1 + \cosh a)}{(\sinh a - a)(\pi^2 + a^2)^3} + \frac{\pi a}{(\pi^2 + a^2)^2(\sinh^2 a - a^2)}, \tag{6.25a}$$

$$B_o^* = \frac{4\pi}{(\pi^2 + a^2)^3}, \tag{6.25b}$$

$$C_o^* = \frac{\pi}{(\pi^2 + a^2)^2} \cdot \frac{(\sinh a \cosh a - a)}{(\sinh^2 a - a^2)} \tag{6.25c}$$

and

$$D_o^* = \frac{4\pi a(1 + \cosh a)}{(\sinh - a)(\pi^2+a^2)^3} - \frac{\pi \sinh^2 a}{(\pi^2 + a^2)^2(\sinh^2 a - a^2)}. \tag{6.25d}$$

Inserting this solution Eq.(6.24) in Eq.(6.16b), multiplying by $\sin(\pi\xi)$, integrating from $\xi = 0$ to $\xi = 1$ and then using Eqs.(6.25a) - (6.25d), we find that

$$T_o = \frac{2(\pi^2 + a^2)^3}{a^2\left[1 - \dfrac{16a\pi^2 \cosh^2(a/2)}{(\pi^2+a^2)^2(a + \sinh a)}\right]}. \tag{6.26}$$

The critical Taylor number for the onset of instability in the absence of modulation is determined by minimizing T_o with respect to a. The result, as stated in Chapter I, is within 1% of the exact numerical answer. The one-mode truncation is thus found to be a very good approximation and can be used for studying the modulated system.

Now, before proceeding further we determine the left vectors satisfying the equations

$$(D^2-a^2)^2\,\tilde{u}_o(\xi) = -T_o a^2 \tilde{v}_o(\xi), \tag{6.27a}$$

$$(D^2-a^2)\tilde{v}_o(\xi) = (1-\xi)\tilde{u}_o(\xi) \tag{6.27b}$$

with the boundary conditions $\tilde{v}_o = \tilde{u}_o = \dfrac{d\tilde{u}_o}{d\xi} = 0$ at $\xi = 0$ and $\xi = 1$.

Again making the ansatz $\tilde{v}_o = \sin\pi\xi$, $\tilde{u}_o(\xi)$ is found from Eq.(6.27a), which is

$$(D^2-a^2)^2\,\tilde{u}_o(\xi) = -T_o a^2 \sin\pi\xi. \tag{6.28}$$

The solution is given by

$$\tilde{u}_o(\xi) = \tilde{A}_o \sinh(a\xi) + \tilde{B}_o \cosh(a\xi) + C_o \sinh(a\xi) + \\ + \tilde{D}_o\,\xi\cosh(a\xi) - \frac{T_o a^2}{(\pi^2+a^2)^2}\sin\pi\xi. \tag{6.29}$$

The boundary conditionson $\tilde{u}_o(\xi)$ yield

$$\tilde{B}_o = \tilde{C}_o = 0\ , \tag{6.30a}$$

$$\tilde{D}_o = \frac{T_o a^2}{(\pi^2+a^2)^2}\ \frac{\sinh a}{(a+\sinh a)} \tag{6.30b}$$

and

$$a\tilde{A}_o = \frac{T_o a^2}{(\pi^2+a^2)^2} - \tilde{D}_o\ . \tag{6.30c}$$

The solvability condition for Eqs.(6.17a) and (6.17b) yields T_1. The condition is expressed as

$$\overline{(\tilde{u}_o\ \ \tilde{v}_o)\begin{pmatrix} \mathrm{Re}\ g(\xi)v_o\,e^{i\omega\tau} \\ -T_1a^2u_o - T_oa^2u_o\mathrm{Re}\ f(\xi)e^{i\omega\tau}\end{pmatrix}} = 0\ , \tag{6.31}$$

where the bar denotes time averaging over the time period $(\bar{T})$ of modulation. The above equation is rewritten as

$$F = 2\int_0^1 \left[\frac{\alpha d}{(\sinh \alpha d)} \cosh\{\alpha d(1-\xi)\}\right] \times \left[A_o^* \sinh(a\xi) + B_o^* \cosh(a\xi)\right.$$

$$+ C_o^* \xi \sinh(a\xi) + D_o^* \xi \cosh(a\xi) + \frac{(1-\xi)\sin h(\pi\xi)}{(\pi^2+a^2)^2} - \frac{4\pi^2}{(\pi^2+a^2)^3} \cos \pi\xi\Big]$$

$$\times \sin(\pi\xi) \,.\, d\xi$$

$$= A_o^* \frac{\alpha d}{\sinh(\alpha d)} \left[\frac{2\pi}{\{(a-\alpha d)^2+\pi^2\}} \sinh(\frac{a+\alpha d}{2}) \cosh(\frac{a-\alpha d}{2}) + \right.$$

$$+ \left.\frac{2\pi}{\{(a+\alpha d)^2+\pi^2\}} \cosh(\frac{a+\alpha d}{2}) \sinh(\frac{a-\alpha d}{2})\right]$$

$$+ B_o^* \left[\frac{\alpha d}{\sinh(\alpha d)} \frac{4\pi(a^2+\alpha^2 d^2+\pi^2)}{\{(a-\alpha d)^2+\pi^2\}\{(a+\alpha d)^2+\pi^2\}}\right.$$

$$\times \cosh(\frac{a+\alpha d}{2}) \cosh(\frac{a-\alpha d}{2})\Big] + C_o^* \frac{\alpha d}{\sinh(\alpha d)}$$

$$\times \left[\frac{2\pi(a^2+\alpha^2 d^2+\pi^2)\sinh a}{\{(a-\pi d)^2+\pi^2\}\{(a+\alpha d)^2+\pi^2\}} - \frac{4\pi(a-\alpha d)}{\{(a-\alpha d)^2+\pi^2\}^2}\right.$$

$$\times \cosh(\frac{a+\alpha d}{2}) \cosh(\frac{a-\alpha d}{2}) - \frac{4\pi(a+\alpha d)}{\{(a+\alpha d)^2+\pi^2\}^2} (\frac{a+\alpha d}{2}) \cosh(\frac{a-\alpha d}{2})\Big]$$

$$+ D_o^* \left[\frac{\alpha d}{\sinh(\alpha d)} \frac{2\pi(a^2+\alpha^2 d^2+\pi^2)\cosh a}{\{(a-\alpha d)^2+\pi^2\}\{(a+\alpha d)^2+\pi^2\}} - \frac{4\pi(a-\alpha d)}{\{(a-\alpha d)^2+\pi^2\}^2}\right.$$

$$\times \sinh(\frac{a+\alpha d}{2}) \cosh(\frac{a-\alpha d}{2}) - \frac{4\pi(a+\alpha d)}{\{(a+\alpha d)^2+\pi^2\}^2} \cosh(\frac{a+\alpha d}{2})$$

$$\sinh(\frac{a-\alpha d}{2})\Big] +$$

$$\frac{1}{\bar{T}} \int_0^{\bar{T}} (\tilde{u}_o \quad \tilde{v}_o) \begin{pmatrix} g(\xi)v_o \; e^{i\omega\tau} \\ -T_1 a^2 u_o - T_o a^2 u_o \; f(\xi) e^{i\omega\tau} \end{pmatrix} d\tau = 0$$

or,

$$< \tilde{u}_o | \; g(\xi)v_o > \frac{1}{\bar{T}} \times \int_0^{\bar{T}} \cos\omega\tau \; d\tau - T_1 a^2 < \tilde{v}_o | u_o > - T_o a^2 < \tilde{v}_o | \; f(\xi) u_o > \times$$

$$\times \frac{1}{\bar{T}} \int_0^{\bar{T}} \cos\omega\tau \; d\tau = 0$$

which leads to $T_1 = 0$.

To solve the $O(\varepsilon)$ equations, we first introduce the Fourier expansion of the functions $g(\xi)v_o$ and $f(\xi)u_o$ as

$$g(\xi)v_o = G \sin(\pi\xi) \tag{6.32a}$$

and

$$f(\xi)u_o = F \sin(\pi\xi) , \tag{6.32b}$$

where

$$G = 2 \int_0^1 g(\xi) \; v_o \sin \pi\xi \, d\xi$$

$$= \frac{4\pi^2}{4\pi^2 + (\alpha \, d)^2} \left[\frac{\cosh(\alpha d) - 1}{\alpha d \; \sinh(\alpha d)} \right] \tag{6.33a}$$

and

$$F = 2 \int_0^1 f(\xi) \; u_o \sin(\pi\xi) \; d\xi \, . \tag{6.33b}$$

Inserting expressions for $f(\xi)$ and $u_o(\xi)$ from Eqs.(6.19b)

$$+ D_o^* \frac{\alpha d}{\sinh(\alpha d)} \left[\frac{2\pi (a^2+\alpha^2 d^2+\pi^2) \cosh a}{\{(a-\alpha d)^2+\pi^2\}\{(a+\alpha d)^2+\pi^2\}} - \frac{4\pi (a-\alpha d)}{\{(a-\alpha d)^2+\pi^2\}^2} \sinh\left(\frac{a+\alpha d}{2}\right) \cosh\left(\frac{a-\alpha d}{2}\right) - \frac{4\pi (a+\alpha d)}{\{(a+\alpha d)^2+\pi^2\}^2} \cosh\left(\frac{a+\alpha d}{2}\right) \sinh\left(\frac{a-\alpha d}{2}\right) \right]$$

$$+ \frac{4\pi^2}{(4\pi^2+\pi^2 d^2)} \quad \frac{1}{(\pi^2 + a^2)^2} \left[1 + \frac{\{1 - \cosh(\alpha d)\}}{\alpha d \sinh(\alpha d)} \right]$$

$$+ \frac{8\pi^3}{(\pi^2 + a^2)^3} \quad \frac{\alpha d \tanh(\alpha d/2)}{(4\pi^2 + \alpha^2 d^2)} \quad . \tag{6.34}$$

For $\omega \to 0$

$$F(\omega = 0) \cong \frac{(\pi^2 + a^2)}{T_o \ a^2} , \tag{6.35}$$

while for $\omega \to \infty$

$$F(\omega) \sim \frac{1.5}{(\alpha d)^3} \tag{6.36}$$

The solutions of Eqs.(6.17a) and (6.17b) proceed by choosing v_1 and u_1 as follows:

$$v_1 = v_1(\xi) \ e^{i\omega\tau} , \tag{6.37a}$$

and

$$u_1 = u_1(\xi) \ e^{i\omega\tau} , \tag{6.37b}$$

with the ansatz $v_1(\xi) = A_1^* \sin(\pi\xi)$. (6.38)

Inserting the above expressions in Eq.(6.17a) and using Eq.(6.22a), we have

$$(D^2-b^2)(D^2-a^2) \ u_1(\xi) = A_1^* \ (1-\xi) \sin(\pi\xi) + G(\omega) \sin \pi\xi , \tag{6.39}$$

where

$$b^2 = a^2 + i\omega \quad . \tag{6.40}$$

The general solution of Eq.(6.39) is

$$u_1(\xi) = P\cosh(a\xi) + Q\sinh(a\xi) + \bar{R}\cosh(b\xi) + S\sinh(b\xi)$$

$$+ \frac{A_1^*(1-\xi)\sin(\pi\xi)}{(\pi^2+a^2)(\pi^2+b^2)} - \frac{2\pi A_1^*(2\pi^2+a^2+b^2)\cos(\pi\xi)}{(\pi^2+a^2)^2(\pi^2+b^2)^2}$$

$$+ \frac{G\sin(\pi\xi)}{(\pi^2+a^2)(\pi^2+b^2)} \qquad (6.41)$$

The boundary conditions on $u_1(\xi)$ are

$$u_1(\xi) = \frac{du_1(\xi)}{d\xi} = 0 \text{ at } \xi = 0 \text{ and } 1.$$

The use of

$u_1(\xi) = 0$ at $\xi = 0$ and $\xi = 1$ give

$$P + \bar{R} = \frac{2A_1^*(2\pi^2+a^2+b^2)}{(\pi^2+a^2)^2(\pi^2+b^2)^2}, \qquad (6.42a)$$

and $P\cosh a + Q\sinh a + \bar{R}\cosh b + S\sinh b$

$$= \frac{-2\pi A_1^*(2\pi^2+a^2+b^2)}{(\pi^2+a^2)(\pi^2+b^2)}; \qquad (6.42b)$$

while that of

$$\frac{du_1(\xi)}{d\xi} = 0 \text{ at } \xi = 0 \text{ and } \xi = 1 \text{ yield}$$

$$aQ + bS = \frac{-\pi(A_1^* + G)}{(\pi^2+a^2)(\pi^2+b^2)} \qquad (6.42c)$$

and

$$aP\sinh a + aQ\cosh a + b\bar{R}\sinh b + bS\cosh b = \frac{\pi G}{(\pi^2+a^2)(\pi^2+b^2)}$$

$$(6.42d)$$

we define

$$X = \frac{2\pi A_1^* (2\pi^2 + a^2 + b^2)}{(\pi^2+a^2)^2 \quad (\pi^2 + b^2)^2} , \qquad (6.43a)$$

$$Y = \frac{\pi (A_1^* + G)}{(\pi^2+a^2)(\pi^2 + b^2)} , \qquad (6.43b)$$

$$Z = \frac{\pi G}{(\pi^2 + a^2)(\pi^2 + b^2)} , \qquad (6.43c)$$

and $\Delta = (a^2+b^2)\sinh a \sinh b + 2ab - 2ab \cosh a \cosh b.$

(6.43d)

Now solving Eqs.(6.42a)-(6.42d), we find

$$P\Delta = X [b^2 \sinh a \sinh b + ab (1 + \cosh b)(1 - \cosh a)]$$

$$+ Y (a \cosh a \sinh b - b \sinh a \cosh b)$$

$$+ Z (a \sinh b - b \sinh a) , \qquad (6.44a)$$

$$Q\Delta = X [ab \sinh a (1+\cosh b) - b^2 \sinh b (1 + \cosh a)]$$

$$+ Y(b \cosh a \cosh b - a \sinh a \sinh b - b)$$

$$+ Z(b \cosh a - b\cosh b) , \qquad (6.44b)$$

$$\bar{R} \Delta = X[a^2 \sinh a \sinh b + ab (1+\cosh a)(1 - \cosh b)]$$

$$+ Y(b \sinh a \cosh b - a \cosh a \sinh b)$$

$$+ Z(b \sinh a - a \sinh b), \qquad (6.44c)$$

$$S\Delta = X[ab \sinh b + ab \sinh b \cosh a - a^2 \sinh a \cosh b - a^2 \sinh a]$$
$$+ Y(a \cosh a \cosh b - b \sinh a \sinh b - a)$$
$$+ Z(a \cosh b - a \cosh a) \,. \qquad (6.44d)$$

Now, using Eqs.(6.32b) in Eq.(6.17b)

$$(D^2 - b^2)\, v_1(\xi) = -T_o a^2\, u_1(\xi) - T_o a^2\, F \sin(\pi\xi)$$

or,

$$A_1^*(\pi^2 + b^2) \sin(\pi\xi) = T_o a^2 [u_1(\xi) + F \sin(\pi\xi)].$$

Multiplying throughout with $\sin(\pi\xi)$ and integrating over $\xi = 0$ to $\xi = 1$

$$\frac{A_1^*(\pi^2+b^2)}{2} = T_o a^2 [\int_0^1 u_1(\xi) \sin(\pi\xi) d\xi + F/2]$$

or, $$A_1^* (\pi^2+b^2) = 2T_o a^2 [\int_0^1 d\xi \sin(\pi\xi) \{ P\cosh(a\xi) + Q\sinh(a\xi) + \bar{R}\cosh(b\xi)$$
$$+ S \sinh(b\xi) + \frac{A_1^*(1-\xi)\sin(\pi\xi)}{(\pi^2+a^2)(\pi^2+b^2)} - \frac{2\pi A_1^*(2\pi^2 + a^2 + b^2)\cos(\pi\xi)}{(\pi^2 + a^2)^2 (\pi^2+b^2)^2}$$
$$+ \frac{G \sin(\pi\xi)}{(\pi^2+a^2)(\pi^2+b^2)} \}] + T_o a^2 F$$

or,

$$A_1^*(\pi^2+b^2) = T_o a^2 + 2T_o a^2 [\frac{\pi P(1 + \cosh a)}{(\pi^2 + a^2)} + \frac{\pi Q \sinh a}{(\pi^2 + a^2)}$$
$$+ \frac{\pi \bar{R}(1 + \cosh b)}{(\pi^2 + b^2)} + \frac{S \sinh a}{(\pi^2 + b^2)}$$
$$+ \frac{A_1^*}{4(\pi^2+a^2)(\pi^2+b^2)} + \frac{G}{2(\pi^2+a^2)(\pi^2+b^2)}].$$

Inserting for P, Q, $\bar{R}$, S and then simplifying

$$A_1^* = \frac{T_o a^2 F}{(\pi^2 + b^2)} + \frac{2T_o a^2}{(\pi^2+b^2)} \left[\frac{\pi^2}{\Delta} \{ a \sinh b(1+\cosh a) - \right.$$

$$- b \sinh a(1 + \cosh b) \} (A_1^*+2G) \times \frac{(b^2 - a^2)}{(\pi^2+a^2)^2(\pi^2+b^2)^2} +$$

$$\left. + \frac{A_1^*}{4(\pi^2+a^2)(\pi^2+b^2)} + \frac{G}{2(\pi^2+a^2)(\pi^2+b^2)} \right]$$

or,

$$A_1^* \left[1 - \frac{T_o a^2}{2(\pi^2 + b^2)} < \frac{1}{(\pi^2+a^2)(\pi^2 + b^2)} + \frac{4\pi^2(b^2-a^2)}{\Delta (\pi^2+a^2)(\pi^2+b^2)^2} \right.$$

$$\left. \times \{ \sinh b (1 + \cosh a) - b \sinh a (1 + \cosh b) \} > \right]$$

$$= \frac{T_o a^2 F}{(\pi^2+b^2)} \left[1 + \frac{G}{F} \quad \frac{1}{(\pi^2+a^2)(\pi^2+b^2)} + \frac{4\pi^2(b^2-a^2)}{\Delta(\pi^2+a^2)^2(\pi^2+b^2)^2} \times \right.$$

$$\left. \times \{ a \sinh b (1+\cosh a) - b \sinh a(1 + \cosh b) \} \right] \tag{6.45}$$

So,

$$A_1^* = \frac{T_o a^2 F}{(\pi^2 + b^2)} \quad \frac{(1 + \frac{G}{F} \mathcal{L})}{1 - \frac{T_o a^2}{2(\pi^2+b^2)} \mathcal{L}}, \tag{6.46}$$

where

$$\mathcal{L} = \frac{1}{(\pi^2+a^2)(\pi^2+b^2)}\left[1 + \frac{4\pi^2(b^2-a^2)}{(\pi^2+a^2)(\pi^2+b^2)} \times \right.$$

$$\left.\{a\sinh b(1+\cosh a) - b\sinh a\,(1+\cosh b)\}\right] \qquad (6.47)$$

We now turn to the $O(\varepsilon^2)$ equations. The solvability condition at this order using Eqs..(6.18a) and (6.18b) with $T_1 = 0$ yields

$$\overline{(\tilde{u}_o \quad \tilde{v}_o)\begin{pmatrix} g(\xi)\ v_1\ e^{i\omega\tau} \\ T_2 a^2 u_o + T_o a^2 u_1 f(\xi) e^{i\omega\tau} \end{pmatrix}} = 0$$

or,

$$T_2 a^2 \langle\tilde{v}_o | u_o\rangle = -\frac{1}{2}\langle\tilde{u}_o | v_1(\xi)\ g(\xi)\rangle - \frac{T_o a^2}{2}\langle\tilde{v}_o |\ f(\xi)\ u_1(\xi)\rangle\ ,$$

$$(6.48)$$

where the factor 1/2 comes from the time averaging.

From Eq.(6. 48) ,

$$T_2 = -\frac{1}{2}\ \mathrm{Re}\left[T_o \frac{\langle\tilde{v}_o | f(\xi)\ u_1(\xi)\rangle}{\langle\tilde{v}_o | u_o\rangle} + \frac{\langle\tilde{u}_o | g(\xi) v_1(\xi)\rangle}{\langle\tilde{v}_o | u_o\rangle a^2}\right]$$

$$(6.49)$$

The explicit form for the fractional correction because of modulation is found to be

$$\frac{T_2}{T_o} = \mathrm{Re}\ [\ \frac{1}{4} \quad \frac{T_o\ a^2\ |F|^2}{<u_o(\xi)|\,u_o(\xi)>\{T_o a^2 - 2\}}$$

$$- \ \frac{1}{2} \quad \frac{F^*\ G\ell(\pi^2 + b^2)}{<u_o(\xi)|u_o(\xi)>\{T_o a^2 \ell - 2(\pi^2+b^2)\}}$$

$$+ \ \frac{T_o a^2}{(\pi^2 + a^2)} \quad \{\frac{F - G\ell}{T_o a^2 \ell - 2(\pi^2 + b^2)}\}\ H^*] \quad ,$$

(6.50)

where

$$H = \int_0^1 f(\xi)\ \tilde{u}_o(\xi)\ d\xi \qquad (6.51)$$

and H^* is its complex conjugate.

6.4 Results in Asymptotic Limits

It is worthwhile to examine the expression in two limits of low and high frequencies.

In the low frequency regime ($\omega \to 0$),

$$f(\xi) \to 1 \quad \text{and} \quad g(\xi) \to 1-\xi.$$

Now,

$$\lim_{\omega \to 0} \frac{<\tilde{u}_o|\,g(\xi)\ v_1(\xi)>}{a^2<\tilde{v}_o|u_o>} = \lim_{\omega \to 0} \frac{\int_0^1 \tilde{u}_o(\xi)\ v_1(\xi)\ d\xi}{a^2 \int_0^1 \tilde{v}_o(\xi)\ u_o(\xi)\ d\xi}$$

$$= \lim_{\omega \to 0} \frac{\int_0^1 (D^2-a^2)\tilde{v}_o(\xi)v_1(\xi)\,d\xi}{a^2 \int_0^1 \tilde{v}_o(\xi)\ u_o(\xi)\ d\xi}$$

$$= - \lim_{\omega \to 0} \frac{\overset{*}{1}}{2}\ T_o \quad . \qquad (6.52)$$

Further

$$\lim_{\omega \to 0} \frac{T_o \langle \tilde{v}_o | f(\xi)\, u_1(\xi) \rangle}{\langle \tilde{v}_o | u_o \rangle} = \lim_{\omega \to 0} \frac{T_o \int_o^1 \tilde{v}_o(\xi)\, u_1(\xi)\, d\xi}{\int_o^1 \tilde{v}_o(\xi)\, u_o(\xi)\, d\xi}$$

$$= - \lim_{\omega \to 0} \frac{\int_o^1 (D^2-b^2) v_1(\xi) u_o(\xi) d\xi + T_o\, a^2 \int_o^1 v_o(\xi) u_o(\xi) d\xi}{a^2 \int_o^1 v_o(\xi)\, u_o(\xi)\, d\xi}$$

$$= \lim_{\omega \to 0} \left[-T_o + \frac{(\pi^2 + b^2)}{(\pi^2 + a^2)} A_1^* T_o \right] . \qquad (6.53)$$

Inserting these equations Eq.(6.52) and (6.53) in Eq.(6.49)

$$T_2(\omega = 0) = \lim_{\omega \to 0} \frac{T_o}{2} \left[1 - \frac{i\omega A_1^*}{(\pi^2 + a^2)} - \frac{A_1^*}{2} \right] . \quad (6.54)$$

Using expression for A_1^* Eq.(6.46) , we arrive at the limits and for $a \simeq 3.12$, we find

$$\frac{T_2(\omega = 0)}{T_o} \simeq -0.10 , \qquad (6.55)$$

which is in good agreement with Hall's[1] value of -0.07 .

In high frequency ($\omega \to \infty$)regime,

$$G/F \to - \frac{1}{(\pi^2 + a^2)^2} \left[\frac{3a}{(\sinh^2 a - a)} \{ \sinh a \cosh a - a \} - \frac{4a^2 (1 + \cosh a)(\sinh a)}{(\pi^2 + a^2)} + \frac{4a \sinh a (a \cosh a - \sinh a)}{(\pi^2 + a^2)} - 1 \right]$$

is a constant.

Also $\mathcal{L} \to 0$, and thus A_1^* has the same form as the corresponding (A_1/A_o) in Eq.(4.41) for the Rayleigh-Bénard problem. The last term in Eq.(6.49) dominates at high frequencies of modulation because $\tilde{v}_o$ and u_1 of the first term and $\tilde{u}_o$ and v_1 of the second have similar structures and G is small than F by a factor $\omega^{1/2}$. This suggests that the threshold shifts in the Couette-Taylor and Rayleigh-Bénard problems have identical structures in the large frequency limit. For $a \simeq 3.12$, we find

$$\frac{T_2}{T_o} \simeq \frac{1.2}{\omega^5} \quad . \tag{6.56}$$

This is in disagreement with Hall's[1] analytic calculation which shows $T_2/T_o \simeq -\frac{1}{\omega^3}$. The difference is not only in the asymptotic power law, but also in sign. It is the latter which shows the qualitative change in stability pattern of the modulated Couette-Taylor flow.

6.5 Discussions

In low frequency regime ($\omega \leq 50$ in units of ν/d^2) the result shows a clear destabilization, and quantitative agreement with Hall[1] is excellent. In high frequency regime we find stabilization —— a qualitatively different result from that of Hall. Even the magnitude of the effect has different asymptotic dependence on the frequency of modulation. This is clear from an observation of the hydrodynamic equations [Eqs.(6.14a) and (6.14b)] for modulated flow. The

$O(\varepsilon)$ term in Eq.(6.14a) is smaller than the $O(\varepsilon)$ term in Eq.(6.14b) by a factor αd. At high frequencies $\alpha d \rightarrow \infty$, and thus the modulation effect will be present in v-equation alone and the structure would be the same as that of corresponding Rayleigh-Bénard problem. And, therefore, stabilization is not unexpected. The same qualitative result is obtained by a Lorenz-like truncated system[9] for CT flow.

Riley and Laurence's numerical procedure is not sensitive enough to find the shift in Taylor number at such high frequencies for low ε. However, for $\varepsilon>1$, they do observe a stabilization at higher frequencies — a result in qualitative agreement with our work. The sudden disappearance of this effect from their result, as ε is lowered, is probably due to the smallness of the magnitude of the effect and the consequent numerical difficulties. As for the experimental studies, Donnelly[6] reports stabilization at all frequencies. The later experiments of Thompson[7] show a low-frequency destabilization but a high frequency stabilization.

Our approach is able to produce the results in the entire range of the frequency of modulation for small ε. Thus it also provides an alternative approach to that of Hall[1] for analyzing modulated flow.

REFERENCES

1. Hall, P., J. Fluid Mech. 67, 29 (1975).
2. Hall, P. and Seminara, G., Proc. R.Soc.Lond.A350,299 (1976).
3. Riley, P.J. and Laurence, R.L., J. Fluid Mech. 75, 626 (1976).
4. Meister, B. and Muzner, W.,Angew, Z., Math. Phys. 17, 537 (1966).
5. Kuhlman, H., Phys. Rev. A32, 1703 (1985).
6. Donnelly, R.J., Proc. Roy. Soc. A281, 130 (1964).
7. Thompson, R., Ph.D. Thesis, Department of Meterology, MIT (1968).
8. Chandrasekhar, S.,"Hydrodynamic and Hydromagnetic Stability"(Clarendon Press, Oxford 1961).
9. Bhattacharjee, J.K., Banerjee, K. and Kumar, K., J. Phys. A19, L835 (1986).

APPENDIX

STROBOSCOPIC MAPS AND CHAOS IN MODULATED BINARY LIQUIDS

A.1 Introduction

A low-order truncation[1-4] of hydrodynamic systems facilitates the handling of non-linearity, while preserving the correspondence with actual hydrodynamic equations in case of double-diffusive systems[3,4]. Agarwal et.al[4]. carried out a Lorenz-like truncation for a system of binary liquids and reproduced hydrodynamic results regarding the convection thresholds. Here, we construct a stroboscopic map[5] associated with a modulated binary liquid near the onset of oscillatory convection. In a suitable coordinate system the only relevant direction of the map is isolated. The existence of a return map in this direction shows the possibility of a the period-doubling route to chaos[6-7]. The threshold for the onset of chaos in the experimentally interesting system of ^3He–^4He is also computed. For simplicity, we work with idealized boundary conditions in this section.

A.2 Hydrodynamic Equations

Lorenz-like truncation for a modulated binary liquid yield the following equations:

$$\dot{x} = \sigma(-x+y+u), \qquad \text{(A.1a)}$$

$$\dot{y} = -xz + r\,(1 + A\cos\omega t)\,x - y - \frac{s\mu^2}{k}\,u, \qquad (A.1b)$$

$$\dot{z} = xy - bz - \frac{\mu^2 s}{k}\,v, \qquad (A.1c)$$

$$\dot{u} = -xv - kr\,(1 + A\cos\omega t)\,x - su - Rsy, \qquad (A.1d)$$

$$\dot{v} = ux - bsv - kbsz\;. \qquad (A.1e)$$

The five modes represent velocity potential, circulating modes of temperature and concentration, and temperature and concentration modes of connective transfer of heat and mass. Here, r is the reduced Rayleigh number R/R_c, with $R_c = \frac{27\pi^4}{4}$; $\mu^2 = \frac{k_T^2}{\chi C_p T}$, where C_p is the specific heat and χ , the susceptibility;A and ω are amplitude and frequency of modulation respectively; b = 8/3. All other symbols have the same meaning is defined in Chapter II.

We now carry out a transformation of variables in Eqs.(A.1a) - (A1.e), which is designed to make the z and v equations in hlinearized form version of Eqs.(A.1a)-(A1.e) independent of each other. The transformation needs

$$\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{v} \end{pmatrix}, \qquad (A.2)$$

where

$$\cos^2\theta = \left[\,1 + \frac{b(1-s)}{\Delta}\,\right] /2 \qquad (A.3)$$

and

$$\Delta^2 = b^2(1-s)^2 + 4\mu^2 s^2 b^2 \qquad (A.4)$$

If we define a vector $\vec{X}$ with components $(X_1, X_2, X_3, X_4, X_5) = (x, y, u, \tilde{z}, \tilde{v})$ then the system of Eqs.(A.1a) - (A1.e) can be rewritten as

$$\dot{X}_1 = \sigma(-X_1 + X_2 + X_3), \qquad (A.5a)$$

$$\dot{X}_2 = r(1+A\cos\omega t)X_1 - X_2 - \frac{\mu^2 s}{k} X_3 - X_1(X_4\cos\theta - X_5\sin\theta) \qquad (A.5b)$$

$$\dot{X}_3 = -kr(1+A\cos\omega t)X_1 - ksX_2 - sX_3$$

$$- \underset{\sim}{\alpha} X_1(X_4\sin\theta + X_5\cos\theta), \qquad (A.5c)$$

$$\dot{X}_4 = -\lambda_1 X_4 + X_1 X_2, \qquad (A.5d)$$

$$\dot{X}_5 = -\lambda_2 X_5 + \frac{X_1 X_3}{\underset{\sim}{\alpha}}, \qquad (A.5e)$$

where

$$\underset{\sim}{\alpha} = k\sqrt{b}/\mu$$

and

$$\lambda_{1,2} = \frac{1}{2}[b(1+s) \pm \Delta] \qquad (A.6)$$

A.3 Perturbation Theory

By setting $r = r_{os}(1+\varepsilon^2)$, where $\varepsilon \ll 1$ and r_{os}, the threshold for the onset of oscillatory convection ($r_{os} < r_s$, the threshold stationary convection), we carry out a perturbative solution for X_1, X_2, X_3, X_4 and X_5. The

system of Eqs.(A.5a) - (A.5e) can be written in compact form as

$$L_{ij}X_j - \varepsilon^2 r_{os} (\delta_{i2} - k\,\delta_{i3})\, X_1 - N_i +$$

$$r_{os}\, A \cos\omega t(\delta_{i2} - k\,\delta_{i3})\, X_1 = 0, \qquad (A.7)$$

where

$$L_{ij} = \begin{pmatrix} \frac{\partial}{\partial t} + \sigma & -\sigma & -\sigma & 0 & 0 \\ -r_{os} & \frac{\partial}{\partial t} + 1 & \frac{\mu^2 s}{k} & 0 & 0 \\ r_{os} & sk & \frac{\partial}{\partial t} + s & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial t} + \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial t} + \lambda_2 \end{pmatrix} \qquad (A.8)$$

N_i is the vector of nonlinear terms having the form

$$N_i = \begin{pmatrix} 0 \\ -X_1X_4\cos\theta + X_1X_5 \sin\theta \\ -X_1X_4\sin\theta + X_1X_5 \cos\theta \\ X_1X_2 \\ \frac{X_1X_3}{\underset{\sim}{\alpha}} \end{pmatrix} \qquad (A.9)$$

and δ_{ij} is Kronecker delta symbol, we work with $\omega \simeq 2\omega_o$, the frequency for parmetric resonance and to apply perturbation theory consistently we set $A = \tilde{A}\,\varepsilon^2$.

We begin by noting that the amplitudes X_i will have the form

$$\vec{X} = \varepsilon W(t) \, \vec{X}^{(o)} \, e^{i\omega_o t} + \varepsilon^2 \vec{X}^{(1)} + \ldots , \qquad (A.10)$$

where $\vec{X}^{(o)}$ is found to be

$$\vec{X}^{(o)} = \begin{pmatrix} 1 \\ \dfrac{r_{os} - (\mu^2 s/k) - (i\omega_o s\mu^2 / k)}{1 - (\mu^2 s/k) - i\,\omega_o} \\ (i\omega_o/\sigma) + 1 - X_2^{(o)} \\ 0 \\ 0 \end{pmatrix} \qquad (A.11)$$

The perturbation theory provides W(t) and higher order terms. The bilinear nonlinearity allows N_i to be expanded as

$$N_i = \varepsilon^2 N_i^{(o)} + \varepsilon^3 N_i^{(1)} + \ldots . \qquad (A.12)$$

Now considering terms up to $O(\varepsilon^2)$, we find that

$\frac{dW}{dt} = 0$ upto the same order in ε and

$$L_{ij} X_j^{(1)} = N_i^{(o)} \quad . \qquad (A.13)$$

The solution is

$$X_j^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(\dfrac{X_2^{(o)} e^{i\omega_o t}}{\lambda_1 + 2i\omega_o}\right) W^2 + \dfrac{2\,\mathrm{Re}\, X_2^{(o)}}{\lambda_1} \, |W|^2 \\ \left(\dfrac{X_2^{(o)} e^{2i\omega_o t}}{\lambda_2 + 2i\omega_o}\right) \dfrac{W^2}{\underset{\sim}{\alpha}} + \dfrac{2\,\mathrm{Re}\, X_3^{(o)}}{\underset{\sim}{\alpha}\lambda_2} \, |W|^2 \end{pmatrix}. \tag{A.14}$$

We now turn to the $O(\varepsilon^3)$ term in Eq. (A.7) to find a relation for determining $W(t)$. Equating the coefficients of $e^{i\omega_o t}$ to zero, we find a condition on W in the form of the differential equation

$$\frac{dW}{d\tau} = (\alpha_1 + i\alpha_2)W - (\beta_1 + i\beta_2)\, W|W|^2$$
$$+ \frac{\tilde{A}}{2}\,(\alpha_1 \div i\,\alpha_2)\, W^* , \tag{A.15}$$

where

$$\tau = \varepsilon^2 t \, . \tag{A.16}$$

$\alpha_1 + i\alpha_2$ and $\beta_1 + i\beta_2$, are obtained from the scalar products

$$\alpha_1 + i\alpha_2 = \langle \vec{X}_L^{(o)} | \vec{M} \rangle / \langle \vec{X}_L^{(o)} | \vec{X} \rangle \tag{A.17}$$

and

$$\beta_1 + i\beta_2 = \langle \vec{X}_L^{(o)} | \vec{N} \rangle / \langle \vec{X}_L^{(o)} | \vec{X} \rangle \tag{A.18}$$

where

$$\vec{X}_L^{(o)} = \left(1, \frac{(i\omega_o + \sigma)}{r_{os}} + k\,\bar{\gamma}\, , \bar{\gamma}\, , 0, 0\right), \tag{A.19}$$

$$\vec{M} = r_{os} \begin{pmatrix} 0 \\ 1 \\ -k \\ 0 \\ 0 \end{pmatrix} , \qquad (A.20)$$

and

$$\vec{N} = \begin{pmatrix} 0 \\ N_1 \mathrm{Cos}\theta + N_2 \mathrm{Sin}\theta \\ \bar{\gamma}(N_1 \mathrm{Cos}\theta - N_2 \mathrm{Cos}\theta) \\ 0 \\ 0 \end{pmatrix} , \qquad (A.21)$$

with

$$\bar{\gamma} = \frac{\sigma - (\mu^2 s/u\, r_{os})(\sigma + i\omega_o)}{s(1 + \mu^2) + i\omega_o} , \qquad (A.22)$$

$$N_1 = \frac{X_2^{(o)}}{\lambda_1 + 2i\omega_o} + 2\,\frac{\mathrm{Re}\, X_2^{(o)}}{\lambda_1} \qquad (A.23)$$

and

$$N_2 = \frac{X_3^{(o)}}{\lambda_2 + 2i\omega_o} + 2\,\frac{\mathrm{Re}\, X_3^{(o)}}{\lambda_2} . \qquad (A.24)$$

A.4. Stroboscopic Map

The concept of discrete map has proven itself worthwhile in analytic study of chaotic behaviour in dynamical systems. Construction of a stroboscopic map[5] is a powerful instrument to study non-autonomous systems which do not admit multiple time perturbation scheme.

A stroboscopic map in an n-dimensional non-autonomous dynamical system is a discrete mapping from an n-dimensional space to other obtained by considering the flow at times separated by regular intervals. We note from Eq.(A.10) that the form of limit cycle is specified by the amplitude W, which is complex. Hence for $\varepsilon << 1$, we can reduce the 5x5 flow system to a 2x2 map.

We begin by taking the form of $W(\tau)$ as

$$W(\tau) = w(\tau)\, e^{i\phi(\tau)} \tag{A.25}$$

where $w(\tau)$, radius of the limit cycle, and $\phi(\tau)$ are real. Inserting Eq. (A.25) in Eq.(A.15) and equating real and imaginary parts separately, we have

$$\dot{w} = \alpha_1 w - \beta_1 w^3 + \delta w \cos 2(\phi - \varphi) \tag{A.26}$$

and

$$\dot{\phi} = \alpha_2 - \beta_2 w^2 + \delta \sin 2(\phi - \varphi) \tag{A.27}$$

where

$$2\delta = a\,[\alpha_1^2 + \alpha_2^2]^{1/2}, \tag{A.28}$$

and

Again the recursion relation for w is obtained by interpolating Eqs.(A.30) and (A.34) using (A.37). It is found to be

$$w_{n+1} = w_n \; e^{(\alpha_1+\delta)T_o} \; [1 + \tilde{\Delta} w_n^2]^{-1/2} \; \cos\left(\frac{\zeta_n}{2}\right) \times$$

$$\times [\, 1 + e^{-4\delta T_o} \tan^2\left(\frac{\zeta_n}{2}\right)]^{1/2} \quad . \qquad (A.38)$$

There are only two independent components of $\vec{X}$ up to $O(\varepsilon)$. We construct two variables as

$$x_n = w_n \cos \phi_n \; ,$$

$$y_n = - \frac{\sigma}{\omega_o} (X_1^{(o)} - X_2^{(o)} - X_3^{(o)})_n = \varepsilon \, w_n \sin\phi_n$$

(A.39)

Using the recursion formulae (A.36) and (A.38) after rotating the coordinate axes by Ψ, we find

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \frac{e^{\alpha_1 T_o}}{1+(p_n^2+q_n^2)\tilde{\Delta}\,]^{1/2}} \begin{pmatrix} e^{\delta T_o} & 0 \\ 0 & e^{-\delta T_o} \end{pmatrix} \begin{pmatrix} \cos\Phi_n & \sin\Phi_n \\ -\sin\Phi_n & \cos\Phi_n \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

(A.40)

where

$$\Phi_n = \frac{\beta_2}{2\beta_1} [\ln (1 + p_n^2 \tilde{\Delta})] \; . \qquad (A.41)$$

After large number of iterations the system (A.40) reduces to one dimensional return map

$$p_{n+1} = p_n \frac{e^{(\alpha_1 + \delta)T_o}}{[1 + p_n^2 \tilde{\Delta}]^{1/2}} \cos\Phi_n \qquad (A.42)$$

Thus, we find that for modulated binary liquid the stroboscopic map reduces to a one-dimensional return map in the $O(\varepsilon)$ analysis. The map is easily seen to have a quadratic top and does show a series of period doubling bifurcations to chaos.

A.5 Numerical Results and Conclusions

As an example, we consider the typical values, $\sigma = 1$, $s = 0.05$, $k = 0.04$ and $\mu \simeq 0.01$ for $^3He-^4He$ mixture at a temperature of about 10 mK from the superfluid transition point. The linear stability analysis yields $r_{os} = 1.12$, $r_s = 6.25$, so that the onset of convection will be oscillatory. The frequency ω_o of the oscillation is 0.14. We set $r = r_{os}(1 + \varepsilon^2)$, where $\varepsilon^2 = 0.01$, that is we are slightly above the threshold for oscillatory convection. The radius of the limit cycle $w_o \simeq 1.0$, as the constants α_1, α_2, β_1 and β_2 turn out to be 0.46, -0.065, and -0.26 respectively. The stroboscopic map of Eq. (A.41) now turns out to be

$$p_{n+1} = \mathcal{R} \cdot p_n \frac{\cos\{\frac{1}{4}\ln(1+0.4)\, p_n^2\}}{\{1 + 0.4\, p_n^2\}^{1/2}}, \qquad \text{(A.43)}$$

where

$$\mathcal{R} = 1.2\, e^{0.36} \qquad \text{(A.44)}$$

This map has a quadratic max at $p_n = p_o = 3.5$ and shows a series of period doubling bifurcations. The limit point

in terms of A [the amplitude of the modulation in Eqs.(A.5b) and (A.5c)] turns out to be A $\simeq$ 0.2.

Numerical integration of the dynamical system for the above parameters shows a sequence of period doubling bifurcations after some precursor transition where the limit cycle of frequency ω_0 changes shapes. These precursor[8] transitions cannot be handled in terms of the stroboscopic map introduced by us in Sec.(A.4). The threshold for chaos in the numerical experiment is A $\simeq$ 0.38, in qualitative agreement with A $\simeq$ 0.2 obtained from the $O(\varepsilon)$ map.

REFERENCES

1. Lorenz, E., J. Atoms. Sci. 20, 130 (1963).

2. Mclaughlin, J.B. and Martin, P.C., Phys. Rev. A12, 186 (1975).

3. DaCosta, L.N., Knobloch, E. and Weiss, N.O., J. Fluid. Mech. 109, 25 (1982).

4. Agarwal, A., Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. B30, 6458 (1984).

5. Broomhead, D., McCreadie, G. and Rowlands, G., Phys. Letts., 84A, 229 (1981).

6. Feigenbaum, M.J., J. Stat. Phys. 19, 25 (1978) and 21, 669 (1979).

7. Bai-Lin, H., Chaos (World Scientific, Singapore, 1984).

8. Kumar, K., Agarwal, A.K., Bhattacharjee, J.K. and Banerjee, K., Phys. Rev. A (in press).

Date Slip

This book is to be returned on the date last stamped.
